Exercise sheet 7
Theoretical Physics 5 : WS 2021/2022
Lecturer : Prof. M. Vanderhaeghen

29.11.2021

Exercise 0.

How much time did it take to complete the task?

Exercise 1. (55 points) : Dirac particle in a scalar potential

Consider a Dirac particle travelling along the z-axis and subject to the scalar square-well potential

$$V(z) = \begin{cases} 0 & \text{when} \quad z < -a/2 & \text{(region I)} \\ V_0 & \text{when} \quad -a/2 \leq z \leq a/2 & \text{(region II)} \\ 0 & \text{when} \quad a/2 < z & \text{(region III)} \end{cases}$$

where $a > 0$ and $V_0 < 0$. In regions I and III, the eigenvalue Dirac equation takes the form

$$\left(\vec{\alpha} \cdot \hat{\vec{p}} c + \beta m_0 c^2\right) \psi = E\psi,$$

while in region II, it has the form

$$\left[\vec{\alpha} \cdot \hat{\vec{p}} c + \beta (m_0 c^2 + V_0)\right] \psi = E\psi.$$

In this second region one can consider that due to the potential, the particle has now an effective mass $m_{\text{eff}} = m_0 + V_0/c^2$.

a) (*10 p.*) Write down the general solution $\psi(z)$ in the three regions with the spin in the z-direction.
   Hints: A plane-wave solution with momentum $\vec{p}$, mass m and spin label s can be written as

$$u(\vec{p}, s) = A \begin{pmatrix} \chi_s \\ \frac{c\ \vec{\sigma}\cdot\vec{p}}{E+mc^2}\, \chi_s \end{pmatrix} e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r} - Et)},$$

   where A is some complex number and χ_s a two-component spinor. For the spin projection along the z-axis, χ_s is an eigenstate of the Pauli matrix σ_3. Do not forget that plane waves can travel in both directions. Remember that the matrices $\vec{\alpha}$ and β in standard representation are given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) *(20 p.)* Impose the continuity condition at $z = \pm a/2$. Express the coefficient of the incoming plane wave of region I in terms of the plane waves of region III. Define for convenience the dimensionless quantity

$$\gamma \equiv \frac{k_1 c}{E + m_0 c^2} \, \frac{E + m_{\rm eff} c^2}{k_2 c},$$

where k_1 is the momentum in regions I and III, and k_2 is the momentum in region II.

c) *(25 p.)* Consider the special case $|m_{\rm eff} c^2| < |E| < m_0 c^2$ corresponding to bound states. Show that these states satisfy

$$k_2 \, \cot\left(\frac{k_2 a}{\hbar}\right) \;=\; -\left(\frac{m_0 V_0}{\kappa_1} + \kappa_1\right),$$

where $\kappa_1 = -i k_1$.
Hints: Show that in the case considered, there could be neither an incoming wave in region I nor an outgoing wave in region III. Show then that continuity requires

$$\mathrm{Im}\left(\frac{1+\gamma}{1-\gamma}\, e^{-\frac{i}{\hbar} k_2 a}\right) \;=\; 0,$$

and that $\gamma = i\Gamma$ is imaginary leading then to

$$\cot\left(\frac{k_2 a}{\hbar}\right) \;=\; \frac{1-\Gamma^2}{2\Gamma}.$$

Exercise 2. (45 points) : Dirac matrices gymnastics

Without using an explicit representation for the Dirac matrices, show that:

a) *(5 p.)* $\gamma_\mu \gamma^\mu = 4$;

b) *(5 p.)* $\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$;

c) *(5 p.)* $\mathrm{Tr}[\not{a}\,\not{b}\,\not{c}\,\not{d}] = 4\,(a_\mu b^\mu c_\nu d^\nu - a_\mu c^\mu b_\nu d^\nu + a_\mu d^\mu b_\nu c^\nu)$, where $\not{a} \equiv \gamma^\mu a_\mu$;

d) *(5 p.)* $\gamma_5^2 = 1$, with $\gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3$;

e) *(5 p.)* $\{\gamma_5, \gamma^\mu\} = 0$;

f) *(5 p.)* $\mathrm{Tr}[\gamma_5] = 0$;

g) *(5 p.)* $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_5] = 0$;

h) *(5 p.)* $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5] = 4i\, \varepsilon^{\mu\nu\rho\sigma}$, where $\varepsilon_{0123} = +1$;

i) *(5 p.)* $\mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] = 0$ if n is odd.