Exercise sheet 6
Theoretical Physics 5 : WS 2021/2022
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (50 points) : Spontaneous symmetry breaking

Consider the following Lagrangian for a complex scalar field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$

$$\mathcal{L} \;=\; (\partial^\mu \phi^*)(\partial_\mu \phi) + \mu^2 \phi^* \phi - \frac{\lambda}{2}\,(\phi^*\phi)^2,$$

Clearly, for $\mu^2 = -m_0^2 c^2/\hbar^2$ and $\lambda = 0$, one recovers the familiar Klein-Gordon Lagrangian. Moreover, this Lagrangian is also invariant under the phase transformation $\phi \to e^{i\alpha}\phi$.

a) (*10 p.*) Treating ϕ and ϕ^* as independent, show that the corresponding Hamiltonian is given by

$$H \;=\; \int d^3x \left(\frac{1}{c^2}\,\dot{\phi}^*\dot{\phi} + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi - \mu^2 \phi^*\phi + \frac{\lambda}{2}\,(\phi^*\phi)^2 \right).$$

Hint: Remember that the conjugate momentum π of a field ϕ is given by $\pi = \partial\mathcal{L}/\partial\dot{\phi}$ and the Hamiltonian by $H = \int d^3x \left(\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}\right)$.

b) (*20 p.*) Find the condition on classical field configurations $\phi_0(x)$ which assures minimization of the energy. Show that there is an infinite set of such configurations and that they are all related by the phase transformation. *Hint*: Try first to minimize the "kinetic" term $\frac{1}{c^2}\,\dot{\phi}^*\dot{\phi} + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi$, then minimize the "potential" term $-\mu^2\phi^*\phi + \frac{\lambda}{2}\,(\phi^*\phi)^2$.

c) (*30 p.*) Suppose that the system is near the minimum $\phi_0 = \mu/\sqrt{\lambda}$. Then it is convenient to define

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}\left[\sigma(x) + i\theta(x)\right],$$

where the fields $\sigma(x)$ and $\theta(x)$ describe (real) fluctuations around the minimum. Rewrite $\mathcal{L}$ in terms of these fluctuations. What are the masses of the σ and θ fields? *Hint*: Remember that the mass can be read off the coefficient quadratic in the field, *e.g.* for a real scalar field Φ with mass m_0 we had the quadratic term $-\frac{1}{2}\frac{m_0^2c^2}{\hbar^2}\Phi^2$.

Exercise 2. (50 points) : Relativistic electrons in a constant magnetic field

To study the behavior of electrons in a constant magnetic field, we have to solve the stationary-state Dirac equation

$$\left(-i\hbar c\ \vec{\alpha}\cdot\vec{D}+\beta m_0c^2\right)\psi = E\psi$$

where we have used minimal substitution $\vec{\nabla}\to\vec{D}\equiv\vec{\nabla}-ie\vec{A}/\hbar c$.

a) (*20 p.*) Verify that

$$\left(\vec{\alpha}\cdot\vec{D}\right)^2 = \vec{D}^2 1+\frac{e}{\hbar c}\vec{\Sigma}\cdot\vec{B}, \qquad \text{where} \qquad \vec{\Sigma}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}.$$

b) (*30 p.*) For the particular case $\vec{A}=(0,xB,0)$ and by considering solutions of the form $\psi = e^{i(p_y y+p_z z)/\hbar}u(x)$, show that the energy eigenvalues E of a relativistic electron in constant magnetic induction $\vec{B}$ are given by

$$E^2 = m_0^2c^4+p_z^2c^2+(2n+1)|eB|\hbar c\pm eB\hbar c, \qquad n\in N.$$

Hint: Remember that the eigenvalues of the harmonic oscillator operator $-\partial_x^2+\omega^2x^2$ are $(2n+1)|\omega|$ with $n\in N$.