Exercise sheet 5
Theoretical Physics 5 : WS 2021/2022
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (40 points) : Real Klein-Gordon field

Using the normal mode expansion of the real Klein-Gordon field

$$\phi(\vec{x},t) \quad = \quad \sum_{\vec{k}} \left(\frac{\hbar c^2}{2\omega_k L^3} \right)^{1/2} \left[a(\vec{k})\, e^{-ik.x} + a^\dagger(\vec{k})\, e^{ik.x} \right]$$

and the equal-time commutation relations

$$\begin{aligned}
&[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0, \\
&[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{x}',t)] = 0, \\
&[\phi(\vec{x},t), \dot{\phi}(\vec{x}',t)] = i\hbar\, c^2\, \delta^{(3)}(\vec{x}-\vec{x}'),
\end{aligned}$$

Show that:

a) (*20 p.*) the creation and annihilation operators satisfy the following commutation relations

$$\begin{aligned}
&[a(\vec{k}), a(\vec{k}')] = 0, \\
&[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0, \\
&[a(\vec{k}), a^\dagger(\vec{k}')] = \delta_{\vec{k},\vec{k}'};
\end{aligned}$$

b) (*10 p.*) the Hamiltonian $H = \int d^3x\, \frac{1}{2} \left[\frac{1}{c^2}\, \dot{\phi}^2 + (\vec{\nabla}\phi)^2 + \mu^2\phi^2 \right]$ takes the form

$$H = \sum_{\vec{k}} \hbar\omega_k \left(a^\dagger(\vec{k}) a(\vec{k}) + \frac{1}{2} \right);$$

c) (*10 p.*) the momentum $\vec{P} = -\int d^3x\, \frac{1}{c^2}\, \dot{\phi}\, \vec{\nabla}\phi$ takes the form

$$\vec{P} = \sum_{\vec{k}} \hbar\vec{k}\; a^\dagger(\vec{k}) a(\vec{k}).$$

Exercise 2. (60 points) : Complex Klein-Gordon field

The complex Klein-Gordon field is used to describe charged bosons. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi, \tag{1}$$

where the field ϕ has the following normal mode expansion

$$\phi(\vec{x}, t) \;=\; \sum_{\vec{k}} \left(\frac{\hbar c^2}{2\omega_k L^3} \right)^{1/2} \left[a(\vec{k})\, e^{-ik\cdot x} + b^\dagger(\vec{k})\, e^{ik\cdot x} \right]$$

and satifies the equal-time commutation relations

$$\begin{aligned} [\phi(\vec{x}, t), \Pi_\phi(\vec{x}', t)] &= i\hbar\, \delta^{(3)}(\vec{x} - \vec{x}'), \\ \left[\phi^\dagger(\vec{x}, t), \Pi_{\phi^\dagger}(\vec{x}', t)\right] &= i\hbar\, \delta^{(3)}(\vec{x} - \vec{x}'), \end{aligned}$$

all other commutators vanishing. In the following, you can conveniently consider the fields ϕ and $\phi^\dagger$ as independent.

a) (*15 p.*) Show that (1) is equivalent to the Lagrangian of two independent real scalar fields with same mass and satisfying the standard equal-time commutation relations.
Hint: Decompose the complex field in real components $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$.

b) (*15 p.*) Write down the conjugate momentum fields Π_ϕ and $\Pi_{\phi^\dagger}$ in terms of ϕ and $\phi^\dagger$, and derive the equal-time commutation relations of a, $a^\dagger$, b and $b^\dagger$.

c) (*15 p.*) Show that (1) is invariant under any global phase transformation of the field $\phi \to \phi' = e^{-i\alpha}\phi$ with α real. Write down the associated conserved Noether current J^μ and express the conserved charge $Q = \int d^3x\, J^0$ in terms of creation and annihilation operators.

d) (*15 p.*) Compute the commutators $[Q, \phi]$ and $[Q, \phi^\dagger]$. Using these commutators and the eigenstates $|q\rangle$ of the charge operator Q, show that the field operators ϕ and $\phi^\dagger$ modify the charge of the system. How would you interpret the operators a, $a^\dagger$, b and $b^\dagger$?