Exercise sheet 4
Theoretical Physics 5 : WS 2021/2022
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (100 points) : Ground state energy of high-density e^- gas in 1^{st} order perturbation theory

The Hamiltonian of a homogeneous electron gas is given by $\hat{H} = \hat{H}_0 + \hat{H}_1$ with

$$\hat{H}_0 = \sum_{\vec{k},s} \frac{\hbar^2 k^2}{2m} a^\dagger_{\vec{k},s} a_{\vec{k},s} \quad \text{and} \quad \hat{H}_1 = \frac{e^2}{2V} \sum_{\vec{k},\vec{p}} \sum_{\vec{q}\neq\vec{0}} \sum_{s,s'} \frac{4\pi}{q^2} a^\dagger_{\vec{k}+\vec{q},s} a^\dagger_{\vec{p}-\vec{q},s'} a_{\vec{p},s'} a_{\vec{k},s}.$$

In the high-density limit, $\hat{H}_1$ is a perturbation to $\hat{H}_0$. Using techniques of perturbation theory, it is possible to estimate in this regime the ground state energy of the interacting electron gas.

a) (*10 p.*) Express the Fermi momentum k_F in terms of the interparticle spacing r_0.
Hint: $\frac{4}{3}\pi r_0^3 = V/N$.

b) (*20 p.*) Determine $\frac{E^{(0)}}{N}$ in terms of k_F.
Hints:

- $E^{(0)} = \langle \Psi_0 | \hat{H}_0 | \Psi_0 \rangle$
- In the limit that the volume of the system becomes infinite, the sums over states can be replaced by integrals:

$$\sum_{\vec{k},s} f_s(\vec{k}) \longrightarrow \frac{V}{(2\pi)^3} \sum_s \int \mathrm{d}\vec{k}\ f_s(\vec{k})$$

c) (*20 p.*) To 1^{st} order of perturbation theory, the energy shift due to the interaction is given by $E^{(1)} = \langle \Psi_0 | \hat{H}_1 | \Psi_0 \rangle$. Show that

$$E^{(1)} = -\frac{4\pi e^2 V}{(2\pi)^6} \int \mathrm{d}\vec{k}\, \theta(k_F - |\vec{k}|) \int \mathrm{d}\vec{q}\, \frac{1}{|\vec{q}|^2}\, \theta(k_F - |\vec{k}+\vec{q}|)$$

Hint: The creation and annihilation operators need to be paired in a way that the matrix element is non-vanishing

d) (*30 p.*) Determine $E^{(1)}/N$ in terms of k_F.
Hint: Changing integration variables $\vec{k} \to \vec{P}$ you could show that the region of integration corresponds to the intersection of two spheres with radii $\vec{P} \pm \vec{q}/2$.

e) (*20 p.*) Express $(E^{(0)} + E^{(1)})/N$ in terms of $r_s \equiv r_0/a_0$ and $a_0 \equiv \hbar^2/me^2$. Evaluate numerically the coefficients and check the validity of the perturbative approach in the high-density limit.