Exercise sheet 2
Theoretical Physics 5 : WS 2021/2022
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (20 points) : Boson wave functions

Write explicitly the following N-boson wave functions in terms of single-boson wave functions $\psi_i(x_j)$

a) *(5 p.)* $\Phi_{3000\cdots0}(x_1, x_2, x_3)$;

b) *(5 p.)* $\Phi_{1101\cdots0}(x_1, x_2, x_3)$;

c) *(5 p.)* $\Phi_{0121\cdots0}(x_1, x_2, x_3, x_4)$;

d) *(5 p.)* $\Phi_{3001\cdots0}(x_1, x_2, x_3, x_4)$;

Exercise 2. (30 Points) : Number of Bosons

Consider the particle number operator $N = \sum_i C_i^\dagger C_i$ for a system of bosons.

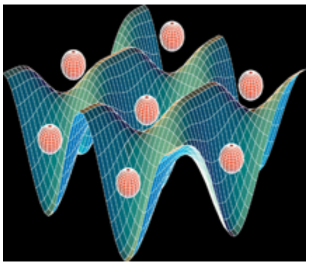
a) *(20 p.)* Show that it commutes with the Hamiltonian

$$H = \sum_{i,j} \langle i|H_0|j\rangle\, C_i^\dagger C_j + \frac{1}{2} \sum_{i,j,k,l} \langle i,j|V|k,l\rangle\, C_i^\dagger C_j^\dagger C_k C_l.$$

What is the physical meaning of this commutation relation?

b) *(5 p.)* Compute $[N, (C_i)^n]$;

c) *(5 p.)* Compute $[N, C_i C_j]$.



Exercise 3. (25 Points) : Bose-Hubbard model

The Bose-Hubbard model gives an approximate description of the physics of interacting bosons on a lattice. It can be used to study systems such as bosonic atoms on an optical lattice, *i.e.* a periodic trap formed by the interference of counterpropagating laser beams. This system resembles a crystal in the sense that the atoms are in a periodic potential. The Hamiltonian of this model is given by (Latin indices refer to lattice sites)

$$H = -t \sum_{<i,j>} \left(C_i^\dagger C_j + C_j^\dagger C_i \right) + \frac{U}{2} \sum_i C_i^\dagger C_i \left(C_i^\dagger C_i - 1 \right),$$

where $< i, j >$ means that the sum is restricted over first neighbors only, and $U > 0$.

a) (*10 p.*) Provide an interpretation of each term of this Hamiltonian;

b) (*10 p.*) Show that in this model, the number of particles is conserved;

c) (*5 p.*) Discuss qualitatively the limits $t \ll U$ and $t \gg U$. What kind of phenomenon can this model reproduce?

Exercise 4. (25 Points) : Bulk Properties of Copper

Let us assume that copper can be described as a non-relativistic free electron gas. The density of copper is 8.96 gm/cm^3, its atomic weight is 63.5 gm/mole, and the number of free electrons per copper atom is well-approximated by 1.

a) (*5 p.*) Calculate the electron Fermi energy for copper in eV. Is it safe to assume that the electrons in copper are non-relativistic?

b) (*5 p.*) Calculate the Fermi temperature for copper, namely the temperature at which the characteristic thermal energy ($k_B T$, where k_B is the Boltzmann constant and T is the Kelvin temperature) equals the Fermi energy for copper. Solid copper has a melting point of 1356 K. Is it safe to assume that the electrons in solid copper are close to the ground state configuration?

c) (*5 p.*) Calculate the degeneracy pressure of copper.

d) (*10 p.*) The bulk modulus of a material measures how it responds to uniform compression:

$$B = -V \frac{\partial P}{\partial V}$$

where V is the volume of the material and P is the pressure. What is B for a free electron gas? How well does the free electron gas model account for the bulk modulus of copper, $13,4 \times 10^{10}$ N/m^2? Is this expected?