The Muon $g - 2$

A sensitive probe for new physics

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PRISMA+ Cluster of Excellence, Institute for Nuclear Physics and Helmholtz Institute Mainz

International School of Subnuclear Physics — IN SEARCH FOR THE UNEXPECTED

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The Quest for New Physics
The Quest for New Physics
The Quest for New Physics

Standard Model does not explain

- Baryon asymmetry
- Mass and scale hierarchies
- Existence of dark matter

Standard Model does not provide a complete description of Nature

- Dark Matter: 83.8%
- Stars, Gas: 15.7%
- Neutrinos: 0.5%
The Quest for New Physics

• Explore the limits of the Standard Model
  • Search for new particles and phenomena at high energies
  • Search for enhancement of rare phenomena
  • Compare precision measurements to SM predictions

• Realise extreme levels of experimental sensitivity, matched by equally precise theoretical calculations

• Control over “hadronic uncertainties” — effects arising from the strong interaction

• Prominent example: anomalous magnetic moment of the muon
Magnetic moment of particles and nuclei

Particle with charge $e$ and mass $m$:

$$\mu = g \frac{e\hbar}{2m} S, \quad S = \frac{\sigma}{2}$$

Pauli equation:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left\{ \frac{1}{2m} \left[ \sigma \cdot (p - eA) \right]^2 + e\Phi \right\} \psi(x, t)$$

$$\leftrightarrow$$

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left\{ \frac{1}{2m} (p - eA)^2 + e\Phi - \frac{e\hbar}{2m} \sigma \cdot B \right\} \psi(x, t)$$

* Non-relativistic limit of Dirac equation: $g = 2$

* Experimental measurement:

$$g_e = 2.0023193 \ldots$$

$$g_\mu = 2.0023318 \ldots$$
Anomalous magnetic moment

* Dirac value of $g = 2$ modified by quantum corrections

$$g = 2(1 + a) \quad \Rightarrow \quad a = \frac{1}{2}(g - 2)$$

* First-order QED correction:

$$a^{(2)} = \frac{\alpha}{2\pi} = 0.00116140 \ldots$$

[J. Schwinger, Phys Rev 73 (1948) 416]

$$a_e^{\exp} = 0.001159652181643(764)$$

$$a_{\mu}^{\exp} = 0.0011659209(6)$$
Higher-order corrections

* QED corrections:

1) \( \gamma\gamma \)
2) \( \mu\gamma \)
3) \( \mu\gamma \)

4) \( \gamma\mu \)
5) \( \gamma\gamma \)
6) \( \gamma\tau \)

\[ QED \text{ corrections:} \]

\[ \gamma\gamma \mu\gamma \]

\[ \mu\gamma \]...

\[ \gamma\mu \]...

\[ \gamma\tau \]...

* Weak corrections:

a) \( \gamma W \)

b) \( \gamma Z \)

\[ \gamma\mu \]

\[ \gamma\nu\mu \]

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Higher-order corrections

* QED corrections:

1) \( \gamma \gamma \mu e \)
2) \( \mu \gamma \gamma e \)
3) \( \mu (vap, e) = \left[ 1 + 13 \ln m_{\mu} - 25 + O(m_{e} m_{\mu}) \right] \left( \alpha \pi \right)^{2} \)

* Weak corrections:

99.994% of \( a_{\mu} \) are due to QED!

* Strong corrections:
**Why do we care?**

* Standard Model estimate of $a_\mu$ deviates from experiment:

$$a_\mu \equiv \frac{1}{2}(g - 2)_\mu = \begin{cases} 116592080(54)(33) \cdot 10^{-11} \\ 116591825(34)(26)(1) \cdot 10^{-11} \end{cases}$$

3.5 $\sigma$

* SM estimate dominated by QED; error dominated by QCD

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{strong}}$$

**Diagram:**

- **QED**
  - Strong
  - QED

- **QED+EW**
  - HLbL
  - HVP
Why do we care?

* Standard Model estimate of $a_\mu$ deviates from experiment:

$$a_\mu \equiv \frac{1}{2}(g - 2)_\mu = \left\{ \begin{array}{l} 116\,592\,080(54)(33) \cdot 10^{-11} \\ 116\,591\,825(34)(26)(1) \cdot 10^{-11} \end{array} \right.$$  

$$\Rightarrow 3.5 \sigma$$

* SM estimate dominated by QED; error dominated by QCD

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{strong}} + a_\mu^{\text{NP?}}$$

<table>
<thead>
<tr>
<th>Hadronic</th>
<th>Hadronic</th>
<th>Contribution from</th>
</tr>
</thead>
</table>

![Diagram of hadronic vacuum polarisation](image1)

![Diagram of hadronic light-by-light scattering](image2)

![Diagram of contribution from “New Physics”](image3)
Why do we care?

* Standard Model estimate of $a_\mu$ deviates from experiment:

$$ a_\mu \equiv \frac{1}{2}(g - 2)_\mu = \begin{cases} 116 592 080(54)(33) \cdot 10^{-11} \\ 116 591 825(34)(26)(1) \cdot 10^{-11} \end{cases} $$

3.5 $\sigma$

* SM estimate dominated by QED; error dominated by QCD

$$ a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{strong}} + a_\mu^{\text{NP}} $$

* New physics effects enhanced by

$$ \delta a_\ell \propto m_\ell^2/M^2 $$

⇒ Muon is more sensitive by a factor

$$ (m_\mu/m_e)^2 \approx 4.3 \cdot 10^4 $$

Contribution from “New Physics”? 

Diagram:
- $\gamma$
- $\mu$
- $\mu$
Why do we care?

* Standard Model estimate of $a_\mu$ deviates from experiment:

$$a_\mu \equiv \frac{1}{2}(g - 2)_\mu = \begin{cases} 116 592 080(54)(33) \cdot 10^{-11} \\ 116 591 825(34)(26)(1) \cdot 10^{-11} \end{cases} \quad \text{3.5 } \sigma$$

* SM estimate dominated by QED; error dominated by QCD

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{strong}} + a_\mu^{\text{NP?}}$$

* Candidates for BSM physics:
  
  - Supersymmetric particles
  
  - Additional gauge bosons:

$$G_{\text{SM}} \longrightarrow G_{\text{SM}} \times U(1)^n$$
Why do we care?

- Standard Model estimate of $a_\mu$ deviates from experiment:
  \[
  a_\mu \equiv \frac{1}{2}(g - 2)_\mu = \begin{cases} 
  116592080(54)(33) \cdot 10^{-11} \\
  116591825(34)(26)(1) \cdot 10^{-11}
  \end{cases}
  \]
  \[\text{\textbullet} 3.5 \sigma\]

- SM estimate dominated by QED; error dominated by QCD

  \[
  a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{strong}} + a_\mu^{\text{NP?}}
  \]

- Candidates for BSM physics:
  - Supersymmetric particles
  - Additional gauge bosons:
    \[
    G_{\text{SM}} \longrightarrow G_{\text{SM}} \times U(1)^n
    \]
    \[
    A' - \text{“dark photon”}
    \]
Experimental determination of $a_\mu$

QED contribution to $a_\mu$

Hadronic contributions to $a_\mu$

Hadronic contributions to $a_\mu$ from lattice QCD
**Measuring $a_{\mu}$ at storage rings**

* Particle with charge $e$ moving in a magnetic field:
  * Momentum turns with cyclotron frequency $\omega_C$
  * Spin turns with $\omega_S$

$$\omega_C = -\frac{eB}{m\gamma}, \quad \omega_S = -g \frac{eB}{2m} - (1 - \gamma) \frac{eB}{m\gamma}$$

⇒ Spin turns relative to the momentum with frequency $\omega_a$

$$\omega_a = \omega_S - \omega_C = -\frac{1}{2}(g - 2) \frac{eB}{m}$$

Measuring $a_\mu$ at storage rings

* Storage rings require vertical focussing — apply electric quadrupole field

$$\omega_a = -\frac{e}{m} \left\{ a_\mu B - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\beta \times E}{c} \right\}$$

* Tune $\gamma$ such that term $\propto (\beta \times E)$ vanishes

“magic” $\gamma$:

$$\gamma_{\text{magic}} = 29.3 \quad \Leftrightarrow \quad p_{\text{magic}} = 3.09 \text{ GeV}/c$$

* Measure two quantities: $\omega_a, B$

[Bargmann, Michel & Telegdi 1959]

BNL experiment E821

Birth: \[ \pi^+ \rightarrow \mu^+ + \nu_\mu \]

Death: \[ \mu^+ \rightarrow e^+ + \bar{\nu}_\mu + \nu_e \]

- Highest energy positrons emitted along muon spin axis
- Select positron above threshold energy

Count rate and wiggle plot:

\[ N(t) = N_0(E) \exp \left( -\frac{t}{\gamma \tau_\mu} \right) \left\{ 1 + A(E) \sin (\omega_\alpha t + \phi(E)) \right\} \]
BNL experiment E821

- Count rate and wiggle plot:

\[ N(t) = N_0(E) \exp\left(-\frac{t}{\gamma\tau_\mu}\right)\left\{1 + A(E) \sin(\omega_a t + \phi(E))\right\} \]

\[ a_\mu^{\text{exp}} = 116\,592\,089\,(54)_{\text{stat}}\,(33)_{\text{syst}} \cdot 10^{-11} \]

From BNL E821 to Fermilab E989

\[ a_{\mu}^{\text{exp}} = 116592089 (54)_{\text{stat}} (33)_{\text{syst}} \cdot 10^{-11} \]

- Total precision of 0.54 ppm, dominated by statistics
- Use hotter beam of Fermilab proton booster: 8 GeV/c
- Suppress pion background — longer pion decay channel
  
  BNL: 80 m  \[ \rightarrow \]  Fermilab: 2 km

- Aim for 100 ppb statistical and 100 ppb systematic error
  
  \[ \rightarrow 0.14 \text{ ppm} \]  total error

- Transport BNL storage ring to Fermilab
Re-assembly of the storage ring
E34 experiment at J-PARC

- ultra-cold muon beam
- measure $a_\mu$ alongside $d_\mu$
- target precision of 0.1 ppm
E34 experiment at J-PARC

\[ \omega = -\frac{e}{m} \left\{ a_\mu B - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\beta \times E}{c} + \frac{\eta}{2} \left( \frac{\beta \times B}{c} + \frac{E}{c} \right) \right\} \]

\[ E = 0 \quad \Rightarrow \quad \omega = -\frac{e}{m} \left\{ a_\mu B + \frac{\eta}{2} \left( \frac{\beta \times B}{c} \right) \right\} \]
QED contribution to $a_\mu$

- QED contribution has been worked out in perturbation theory to 5th order in $\alpha$

- Order $\alpha$ (1-loop)

- Order $\alpha^2$ (2-loop)
Complete Tenth-Order QED Contribution to the Muon $g - 2$

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2Nishina Center, RIKEN, Wako, Japan 351-0198
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4Laboratory for Elementary Particle Physics, Cornell University, Ithaca, New York 14853, USA

(Received 24 May 2012; published 13 September 2012)
QED contribution to $\alpha_\mu$

\begin{align*}
I(a) & & I(b) & & I(c) & & I(d) & & I(e) \\
I(f) & & I(g) & & I(h) & & I(i) & & I(j) \\
II(a) & & II(b) & & II(c) & & II(d) & & II(e) \\
II(f) & & III(a) & & III(b) & & III(c) & & IV \\
V & & VI(a) & & VI(b) & & VI(c) & & VI(d) \\
VI(f) & & VI(g) & & VI(h) & & VI(i) & & VI(j) & & VI(k)
\end{align*}
QED contribution to $a_\mu$


High-precision calculation of the 4-loop contribution to the electron $g$-2 in QED

Stefano Laporta

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A B S T R A C T

I have evaluated up to 1100 digits of precision the contribution of the 891 4-loop Feynman diagrams contributing to the electron $g$-2 in QED. The total mass-independent 4-loop contribution is

$$a_e = -1.912245764926445574152647167439830054060873390658725345 \cdots \left( \frac{\alpha}{\pi} \right)^4.$$
Theory confronts experiment

![Diagram showing experimental sensitivities compared to SM predictions](Jegerlehner, arXiv:1705.00263)

- Experimental sensitivity of E989 exceeds total theory uncertainty by far!
Hadronic contributions to $a_\mu$

Hadronic vacuum polarisation:

\[
\gamma \quad \mu \quad \mu
\]

\[a_\mu^{\text{hvp}} = (6933 \pm 25) \cdot 10^{-11}\]

Dispersion theory:

(combined $e^+e^-$ data)

[Keshavarzi et al., arXiv:1802.02995]

Hadronic light-by-light scattering:

\[
\gamma \quad \mu \quad \mu
\]

\[a_\mu^{\text{hlbl}} = (105 \pm 26) \cdot 10^{-11}\]

Model estimates:

“Glasgow consensus”

[Prades, de Rafael, Vainshtein 2009]
Hadronic vacuum polarisation

* Hadronic electromagnetic current:

\[
J^\mu(x) = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s + \frac{2}{3}\bar{c}\gamma^\mu c + \ldots
\]

\[
(q^\mu q^\nu - q^2 g^\mu\nu) \Pi(q^2) = ie^2 \int d^4x e^{iq\cdot x} \langle 0 | T J^\mu(x)J^\nu(0) | 0 \rangle
\]

* Optical theorem:

\[
a_{\text{hvp}}^{\mu} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_0^2}^\infty ds \frac{R_{\text{had}}(s) \hat{K}(s)}{s^2}, \quad R_{\text{had}}(s) = \sigma(e^+e^- \rightarrow \text{hadrons}) \frac{4\pi \alpha(s)}{(3s)}
\]

\[
2 \text{Im} \begin{array}{c}
\end{array} = \sum_{\text{had}} \int d\Phi \left| \begin{array}{c}
\end{array} \right|^2 \propto \sigma(e^+e^- \rightarrow \text{hadrons})
\]

\[
= \int \frac{ds}{\pi(s-q^2)} \text{Im} \begin{array}{c}
\end{array}
\]
HVP contribution from dispersion relations

* Knowledge of $R_{\text{had}}(s)$ required down to pion threshold

\[
a^\text{hvp}_\mu = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left\{ \int_{m^2_{\pi^0}}^{E^2_{\text{cut}}} ds \frac{R_{\text{had}}(s) \hat{K}(s)}{s^2} + \int_{E^2_{\text{cut}}}^{\infty} ds \frac{R^{\text{PQCD}}_{\text{had}}(s) \hat{K}(s)}{s^2} \right\}
\]

⇒ Use experimental data for cross section ratio $R_{\text{had}}(s)$

Initial state radiation (ISR) vs. beam energy tuning:

$s = M^2_\phi$, $s' = s(1 - k)$, $k = E_\gamma/E_{\text{beam}}$

[BESIII Collaboration, 2016]  [Jegerlehner, arXiv:1705.00263]
HVP contribution from dispersion relations

Knowledge of $R_{\text{had}}(s)$ required down to pion threshold

$$ a_{\mu}^{\text{hvp}} = \left(\frac{\alpha m_{\mu}}{3\pi}\right)^2 \left\{ \int_{m_{\pi}^2}^{E_{\text{cut}}^2} ds \frac{R_{\text{had}}(s) \hat{K}(s)}{s^2} + \int_{E_{\text{cut}}^2}^{\infty} ds \frac{R_{\text{had}}^{\text{PQCD}}(s) \hat{K}(s)}{s^2} \right\} $$

⇒ Use experimental data for hadronic cross section $R_{\text{had}}(s)$

Low-energy region dominates

[Jegerlehner, arXiv:1705.00263]
HVP contribution from dispersion relations

- Tension of $\approx 3.7$ standard deviations between SM and experiment
- Overall precision of HVP estimate: $\approx 0.4\%$
- Theory estimate subject to experimental uncertainties
- Disagreement over individual hadronic channels

[Keshavarzi et al., arXiv:1802.02995]
Hadronic Light-by-Light scattering

* No simple dispersive framework
* Identify dominant sub-processes, e.g.

\[ \gamma, \pi^0, \eta, \eta', \ldots \]

* Individual contributions estimated using model calculations
* Dispersive formalism set up for various subprocesses
  
  [Colangelo et al., Pauk & Vanderhaeghen, 2014 ff]

* Other approaches: functional methods, lattice QCD
Hadronic Light-by-Light scattering

**Dominant hadronic contributions to** $a^\text{hlbl}_\mu$ 

[Nyffeler, arXiv:1710.09742]

\begin{equation}
\begin{array}{c}
\begin{aligned}
\mu^-(p) & \quad \mu^-(p') \\
\downarrow k = p' - p \\
\pi^+ & \quad \pi^0, \eta, \eta' \\
\text{pion-loop (dressed)} & \quad \text{pseudoscalar exchanges} \\
\end{aligned}
\end{array}
\end{equation}

Contribution: Chiral counting: $N_c$ counting:

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BPP</th>
<th>HKS, HK</th>
<th>KN</th>
<th>MV</th>
<th>BP, MdRR</th>
<th>PdRV</th>
<th>N, JN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0$, $\eta$, $\eta'$</td>
<td>85±13</td>
<td>82.7±6.4</td>
<td>83±12</td>
<td>114±10</td>
<td>-</td>
<td>114±13</td>
<td>99±16</td>
</tr>
<tr>
<td>axial vectors</td>
<td>2.5±1.0</td>
<td>1.7±1.7</td>
<td>-</td>
<td>22±5</td>
<td>-</td>
<td>15±10</td>
<td>22±5</td>
</tr>
<tr>
<td>scalars</td>
<td>-6.8±2.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-7±7</td>
<td>-7±2</td>
</tr>
<tr>
<td>$\pi$, $K$ loops</td>
<td>-19±13</td>
<td>-4.5±8.1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-19±19</td>
<td>-19±13</td>
</tr>
<tr>
<td>$\pi$, $K$ loops + subl. $N_C$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0±10</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>quark loops</td>
<td>21±3</td>
<td>9.7±11.1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.3 (c-quark)</td>
<td>21±3</td>
</tr>
<tr>
<td>Total</td>
<td>83±32</td>
<td>89.6±15.4</td>
<td>80±40</td>
<td>136±25</td>
<td>110±40</td>
<td>105±26</td>
<td>116±39</td>
</tr>
</tbody>
</table>

BPP = Bijnens, Pallante, Prades ’95, ’96, ’02; HKS = Hayakawa, Kinoshita, Sanda ’95, ’96; HK = Hayakawa, Kinoshita ’98, ’02; KN = Knecht, AN ’02; MV = Melnikov, Vainshtein ’04; BP = Bijnens, Prades ’07; MdRR = Miller, de Rafael, Roberts ’07; PdRV = Prades, de Rafael, Vainshtein ’09; N = AN ’09, JN = Jegerlehner, AN ’09
The muon $g - 2$ in lattice QCD
Hadronic contributions in precision observables

- Accuracy of Standard Model tests limited by hadronic contributions
- Employ “ab initio” approach: Lattice QCD

```
gluon quark
```

“Clover” @ Mainz
Outline

Basic concepts of Lattice QCD

Hadronic vacuum polarisation from lattice QCD

Hadronic light-by-light scattering

Summary & Outlook
Beyond Perturbation Theory: Lattice QCD

“Lattice QCD is the non-perturbative treatment of the gauge theory of strong interactions, based on regularised, Euclidean functional integrals.”

Minkowski space-time, continuum \(\longrightarrow\) Euclidean space-time, discretised

\[
\text{Lattice spacing:}\quad a,\quad x_\mu = n_\mu a,\quad a^{-1} = \Lambda_{\text{UV}}
\]

\[
\text{Finite volume:}\quad L^3 \cdot T,\quad N_s = L/a,\quad N_t = T/a
\]

(anti)quarks: \(\bar{\psi}(x), \psi(x)\)

lattice sites

Gluons: \(U_\mu(x) \equiv e^{aA_\mu(x)} \in \text{SU}(3)\)

links

Field tensor:
\[
U_\mu(x)U_\nu(x + \hat{\mu})U_\mu(x + \hat{\nu})^\dagger U_\nu(x)^\dagger
\]

“plaquettes”
Beyond Perturbation Theory: Lattice QCD

* Formulate a lattice action in terms of quark fields, link variables and plaquettes:

\[ S_{\text{lat}}[U, \bar{\psi}, \psi] = S_{\text{G}}[U] + S_{\text{F}}[U, \bar{\psi}, \psi] \]

* Reproduce Euclidean continuum action as \( a \to 0 \)

\[ S = -\frac{1}{2g_0^2} \int d^4 x \ Tr F_{\mu\nu}(x)^2 + \int d^4 x \ \sum_{f=u,d,s,...} \bar{\psi}_f(x)(\not{D} + m_f)\psi(x) \]

* Discretisation is not unique! Widely used fermionic actions:

<table>
<thead>
<tr>
<th>Staggered</th>
<th>Wilson</th>
<th>Domain wall</th>
<th>Overlap</th>
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<tr>
<td>HISQ</td>
<td>O((a)) improved Wilson</td>
<td>Fixed point</td>
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</tr>
<tr>
<td></td>
<td>twisted-mass Wilson</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Beyond Perturbation Theory: Lattice QCD

Lattice formulation ...

... preserves gauge invariance

... defines observables without reference to perturbation theory

... allows for stochastic evaluation of observables

\[
\langle \Omega \rangle = \frac{1}{Z} \int D[U] D[\bar{\psi}, \psi] \Omega \ e^{-S_G[U] - S_F[U, \bar{\psi}, \psi]}
\]

\[
= \frac{1}{Z} \int D[U] \Omega \ \prod_{f=u,d,s,...} \det(\mathcal{D} + m_f) \ e^{-S_G[U]}
\]

\[
= \frac{1}{Z} \int \prod_{x,\mu} dU_\mu(x) \Omega \ \prod_{f=u,d,s,...} \det(\mathcal{D}^{\text{lat}} + m_f) \ e^{-S_G^{\text{lat}}[U]}
\]
Monte Carlo Simulation

1. Generate set of $N_c$ configurations of gauge fields $\{U^\mu(x)\}_i, \ i = 1, \ldots N_c$ with probability distribution

$$W = \prod_{f=u,d,s,\ldots} \det(D^\text{lat} + m_f) \ e^{-S^\text{lat}_G[U]}$$

Define an algorithm based on a Markov process

Generate sequence: $\{U\}_1 \rightarrow \{U\}_2 \rightarrow \ldots \rightarrow \{U\}_{N_c}$

Transition probability given by $W$ "Importance sampling"

Strong growth of numerical cost near physical $m_u, m_d$

Pion mass, i.e. lightest mass in pseudoscalar channel:

$$\approx 500 \text{ MeV} \quad \longrightarrow \quad \approx 130 \text{ MeV}$$

(2001) \quad (\geq 2015)
Monte Carlo Simulation

1. Generate set of $N_c$ configurations of gauge fields $\{U_\mu(x)\}_i$, $i = 1, \ldots N_c$ with probability distribution

$$W = \prod_{f=u,d,s,\ldots} \det(\mathcal{P}_{\text{lat}} + m_f) e^{-S_{\text{G}}[U]}$$

Define an algorithm based on a Markov process

Generate sequence: $\{U\}_1 \rightarrow \{U\}_2 \rightarrow \ldots \rightarrow \{U\}_{N_c}$

Transition probability given by $W$ “Importance sampling”

2. Evaluate observable on each configuration:

$$\bar{\Omega} = \frac{1}{N_c} \sum_{i=1}^{N_c} \Omega_i, \quad \langle \Omega \rangle = \lim_{N_c \to \infty} \bar{\Omega}, \quad \text{statistical error } \propto 1/\sqrt{N_c}$$

3. Repeat 1. and 2. for different lattice spacings, quark masses and volumes
Correlation functions

* Spectral information contained in correlation functions

\[
\sum_{x,y} e^{i p \cdot (y-x)} \langle O_{\text{had}}(y) O_{\text{had}}^\dagger(x) \rangle = \sum_n w_n(p) e^{-E_n(p)(y_0-x_0)}
\]

\[
\text{(}y_0-x_0) \to \infty \quad \Rightarrow \quad w_1(p) e^{-E_1(p)(y_0-x_0)}
\]

* Ground state dominates at large distances

* \(O_{\text{had}}(x)\): interpolating operator

Pion: \(O_\pi = \bar{u} \gamma_5 d, \; \bar{u} \gamma_0 \gamma_5 d\)

\(\rho\)-meson: \(O_\rho = \bar{u} \gamma_k d\)

Nucleon: \(O_N = \epsilon_{abc} (u^a C \gamma_5 d^b) u^c\)

\[\rightarrow\] projects on all states with the same quantum numbers
Correlation functions

* Spectral information contained in correlation functions

\[ \sum_{x,y} e^{ip \cdot (y-x)} \left\langle O_{\text{had}}(y) \, O_{\text{had}}^\dagger(x) \right\rangle = \sum_n w_n(p) \, e^{-E_n(p)(y_0-x_0)} \]

\[ (y_0-x_0) \to \infty \quad \Rightarrow \quad w_1(p) \, e^{-E_1(p)(y_0-x_0)} \]

* Ground state dominates at large distances

Nucleon “effective mass”

Strong growth of statistical noise as \( t \to \infty \)

[Capitani et al., arXiv:1504.04628]
Systematic effects

*Lattice artefacts:*

\[ \left( \frac{m_N}{f_\pi} \right)^{\text{lat}} = \left( \frac{m_N}{f_\pi} \right)^{\text{cont}} + O(a^p), \quad p \geq 1 \]

→ extrapolate to continuum limit from \( a \approx 0.05 - 0.12 \text{ fm} \)

*Finite volume effects*

- Empirically: \( m_\pi L \geq 4 \) sufficient for many purposes
- Provide information on \textit{scattering phase shifts}

*Unphysical quark masses*

- Chiral extrapolation to physical values of \( m_u, m_d \)

*Inefficient sampling of SU(3) group manifold*

- Simulations become trapped in topological sectors as \( a \to 0 \)
- Use \textit{open boundary conditions} in time direction \[\text{[Lüscher & Schaefer, 2012]}\]
Global analyses of Lattice QCD results

FLAG Report:

- Performs PDG-style global analyses and averages
  - SM parameters (quark masses, $\alpha_s$)
  - Flavour physics ($K$, $D$, $B$ mesons)
  - Low-energy constants
  - Nucleon matrix elements

- Example: strong coupling

\[ \alpha^{(5)}_{\text{MS}}(M^2_Z) = \begin{cases} 
0.11823(81) & \text{FLAG 2019} \\
0.1174(16) & \text{PDG 2018} 
\end{cases} \]

- Combination yields

\[ \alpha^{(5)}_{\text{MS}}(M^2_Z) = 0.11806(72) \]

[S. Aoki et al., arXiv:1902.08191]
Reminder: hadronic contributions to $a_\mu$

Hadronic vacuum polarisation:

\[ a_\mu^{\text{hvp}} = (6933 \pm 25) \cdot 10^{-11} \]

(combined $e^+e^-$ data)

[Keshavarzi et al., arXiv:1802.02995]

Dispersion theory:

Hadronic light-by-light scattering:

\[ a_\mu^{\text{hlbl}} = (105 \pm 26) \cdot 10^{-11} \]

"Glasgow consensus"

[Prades, de Rafael, Vainshtein 2009]
Motivation for first-principles approach:

* No reliance on experimental data
  — except for simple hadronic quantities to fix bare parameters

* No model dependence
  — except for chiral extrapolation and constraining the IR regime

* Can lattice QCD deliver estimates with sufficient accuracy in the coming years?
  \[
  \frac{\delta a_{\mu}^{\text{hvp}}}{a_{\mu}^{\text{hvp}}} < 0.5\%, \quad \frac{\delta a_{\mu}^{\text{hlbl}}}{a_{\mu}^{\text{hlbl}}} \lesssim 10\%
  \]
The muon g – 2 in lattice QCD

Progress in Particle and Nuclear Physics

Volume 104, January 2019, Pages 46-96

Review

Lattice QCD and the anomalous magnetic moment of the muon

Harvey B. Meyer, Hartmut Wittig

Show more

https://doi.org/10.1016/j.ppnp.2018.09.001

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arXiv:1807.09370
The Mainz \((g - 2)_\mu\) Lattice QCD project

Collaborators:


N. Asmussen, J. Green, G. Herdoíza, B. Hörz

- Direct determinations of LO \(a_\mu^{hvp}\)
- Running of \(\alpha\) and \(\sin^2\theta_W\)
- Exact QED kernel
- Forward scattering amplitude
- Transition form factor for \(\pi^0 \rightarrow \gamma^*\gamma^*\)

Lattice QCD approach to HVP

* Convolution integral over Euclidean momenta: 

$$d_{\mu}^{\text{hvp}} = \left( \frac{\alpha}{\pi} \right)^2 \int_0^{\infty} dQ^2 \ f(Q^2) \hat{\Pi}(Q^2), \quad \hat{\Pi}(Q^2) = 4\pi^2 \left( \Pi(Q^2) - \Pi(0) \right)$$

* Vacuum polarisation tensor:

$$\Pi_{\mu\nu}(Q) = i \int d^4x \ e^{iQ \cdot (x-y)} \left\langle J_{\mu}(x)J_{\nu}(y) \right\rangle \equiv (Q_{\mu}Q_{\nu} - \delta_{\mu\nu}Q^2)\Pi(Q^2)$$

* Electromagnetic current:

$$J_{\mu} = \frac{2}{3} \bar{u} \gamma_{\mu} u - \frac{1}{3} \bar{d} \gamma_{\mu} d - \frac{1}{3} \bar{s} \gamma_{\mu} s + \ldots$$

* Weight function $f(Q^2)$ strongly peaked near muon mass
Lattice QCD approach to HVP

* Convolution integral over Euclidean momenta:

\[
\alpha_{\mu}^{\text{hvp}} = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dQ^2 f(Q^2) \hat{\Pi}(Q^2), \quad \hat{\Pi}(Q^2) = 4\pi^2 \left( \Pi(Q^2) - \Pi(0) \right)
\]

* Weight function \( f(Q^2) \) strongly peaked near muon mass

\[
Q_{\min}^2 = (2\pi/L)^2, \quad m_{\pi}^{\text{phys}} L \approx 4 \implies L \approx 6 \text{ fm}
\]
**Lattice QCD approach to HVP**

* Time-momentum representation:

\[
a^\text{hvp}_\mu = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dx_0 \, \tilde{K}(x_0) \, G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle
\]

\[
\tilde{K}(x_0) = 4\pi^2 \int_0^\infty dQ^2 \, f(Q^2) \left[ x_0^2 - \frac{4}{Q^2} \sin^2 \left( \frac{1}{2} Q x_0 \right) \right]
\]

- Significant contribution from tail of \( G(x_0) \)
- Exponentially increasing noise-to-signal ratio:

\[
R_{NS} \propto \exp \{ (m_V - m_\pi) x_0 \}
\]
Lattice QCD approach to HVP

* Time-momentum representation:

\[ a_{\mu}^{\text{hvp}} = \left( \frac{e}{\pi} \right)^2 \int_{0}^{\infty} dx_0 \tilde{K}(x_0) G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle \]

\[ \tilde{K}(x_0) = 4\pi^2 \int_{0}^{\infty} dQ^2 f(Q^2) \left[ x_0^2 - \frac{4}{Q^2} \sin^2 \left( \frac{1}{2} Q x_0 \right) \right] \]

* Control long-distance behaviour of \( G(x_0) \) — large statistical noise

\[ G(x_0) = \begin{cases} 
G(x_0)_{\text{data}}, & x_0 \leq x_{0,\text{cut}} \\
G(x_0)_{\text{ext}}, & x_0 > x_{0,\text{cut}} 
\end{cases} \]

* \( G(x_0) \) dominated by two-pion state for \( x_0 \to \infty \)
Challenges:

* Statistical accuracy at the sub-percent level required
* Control infrared regime of vector correlator: $G(x_0)$ at large $x_0$
* Perform comprehensive study of finite-volume effects
* Include quark-disconnected diagrams
* Include isospin breaking: $m_u \neq m_d$, QED corrections
Simulations and Machines

- JUQUEEN
- Clover
- MOGON II
- Hazel Hen
Gauge ensembles

* $N_f = 2 + 1$ flavours of $O(a)$ improved Wilson quarks

* Four values of the lattice spacing: $a = 0.085, 0.077, 0.065, 0.050$ fm

* Pion masses and volumes:

  $m_{\pi}^{\text{min}} \approx 135$ MeV, $m_{\pi}L > 4$

* Check of finite-volume effects
Controlling the infrared regime

* TMR integrand and its long-distance behaviour:

\[ a_{\mu}^{\text{hvp}} = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dx_0 \tilde{K}(x_0) G(x_0), \quad G(x_0) = -a^3 \sum \langle J_k(x)J_k(0) \rangle \]

* Large-\(x_0\) regime still statistics-limited for \(x_0 \gtrsim 2.5\) fm

\[ m_\pi = 200\ \text{MeV} \quad \text{and} \quad m_\pi = 130\ \text{MeV} \]

[Gérardin et al., arXiv:1904.03120]
Isovector channel

* Saturation of the isovector correlator by low-lying states

\[ G(x_0) = \sum_{n=1}^{\infty} A_n e^{-\omega_n x_0} \quad \text{as} \quad x_0 \to \infty \quad \omega_n = 2 \sqrt{m_\pi^2 + k^2} \]
* Saturation of the isovector correlator by low-lying states

\[ G(x_0) = \sum_{n=1}^{\infty} A_n e^{-\omega_n x_0} \quad \text{as} \quad x_0 \to \infty \quad \omega_n = 2 \sqrt{m_\pi^2 + k^2} \]
Finite-volume effects

* Finite-volume correction

\[ a_{\mu}^{hvp}(\infty) - a_{\mu}^{hvp}(L) = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dx_0 \tilde{K}(x_0) [G(x_0, \infty) - G(x_0, L)] \]

* Finite volume:

\[ G^{pp}(x_0, L) \xrightarrow{x_0 \to \infty} \sum_n |A_n|^2 e^{-\omega_n x_0}, \quad |A_n|^2 = \frac{2k^5}{3\pi\omega^2} \frac{|F_\pi(\omega)|^2}{\{k\phi'(k) + k\delta'(k)\}} \]

* Iso-vector correlator in infinite volume

\[ G^{pp}(x_0, \infty) = \int_0^\infty d\omega \omega^2 \rho(\omega^2) e^{-\omega|x_0|}, \quad \rho(\omega^2) = \frac{1}{48\pi^2} \left( 1 - \frac{4m_\pi^2}{\omega^2} \right)^{3/2} |F_\pi(\omega)|^2 \]

* Timeline pion form factor: \( F_\pi(\omega) \)

[Gérardin et al., arXiv:1904.03120]
Finite-volume effects

* Dedicated study for $m_\pi = 280\text{ MeV}$ (H105, N101: $m_\pi L = 3.8, 5.8$)

\[ \frac{5}{9} G_i(t) \tilde{K}(t)/m_\mu \]

$L = 2.8\text{ fm}$
$L = 2.8\text{ fm} + \text{FSE}$
$L = 4.1\text{ fm}$

[Gérardin et al., arXiv:1904.03120]
Finite-volume effects

* Dedicated study for $m_\pi = 280\text{ MeV}$ (H105, N101: $m_\pi L = 3.8, 5.8$)

\[
L = 2.8\text{ fm} \\
L = 2.8\text{ fm} \\
L = 4.1\text{ fm}
\]

* Finite-size effects well described by parameterisation of $F_\pi(\omega)$

[Gérardin et al., arXiv:1904.03120]
Extrapolation to the physical point

* Contributions from light, strange and charm quarks

\[ (a_\mu^{\text{hvp},c}) = 14.66(45)(6) \times 10^{-10} \]
\[ (a_\mu^{\text{hvp},s}) = 54.5(2.4)(0.6) \times 10^{-10} \]
\[ (a_\mu^{\text{hvp},ud}) = 674(12)(5) \times 10^{-10} \]

Error limited by lattice scale

[Gérardin et al., arXiv:1904.03120]
Extrapolation to the physical point

* Contributions from light, strange and charm quarks

\[(a_{\mu}^{\text{hvp}})^c = 14.66(45)(6) \times 10^{-10}\]
\[(a_{\mu}^{\text{hvp}})^s = 54.5(2.4)(0.6) \times 10^{-10}\]
\[(a_{\mu}^{\text{hvp}})^{ud} = 674(12)(5) \times 10^{-10}\]

Error limited by lattice scale

[Gérardin et al., arXiv:1904.03120]
Towards sub-percent accuracy

- Quark-disconnected diagrams

- Isospin breaking corrections

\[ m_u/m_d = 0.46(2)(2) \]
\[ q_u = 2/3, \quad q_d = -1/3 \]
Isospin breaking effects

* Simulate QCD+QED for $m_u \neq m_d$  

\[
\langle \Omega \rangle = \frac{1}{Z} \int D[A] D[U] D[\bar{\psi}, \psi] \ \Omega \ e^{-S_{\gamma}[A]-S_{G}[U]-S_{F}[A,U,\bar{\psi},\psi]}
\]

Generate $U(1)$ gauge field stochastically and multiply $SU(3)$ link variables:

\[
U_\mu(x) \rightarrow e^{ieA_\mu(x)} U_\mu(x)
\]

( electro-quenched)

* Expand correlator in powers of $(m_d - m_u)$ and $\alpha = e^2 / 4\pi$

Results and comparison

* Final result at the physical point

\[ a_\mu^{\text{hvp}} = (720 \pm 12.4_{\text{stat}} \pm 9.9_{\text{syst}}) \times 10^{-10} \]

(total uncertainty: 2.2%)

* Individual contributions:
Results and comparison

* Lattice QCD vs. dispersion theory:

Our result:

\[ a_{\mu}^{hvp} = (720.0 \pm 15.8) \cdot 10^{-10} \]

Dispersion theory:

\[ (a_{\mu}^{hvp})_{\text{disp}} = (693.3 \pm 2.5) \cdot 10^{-10} \]

Global lattice average:

\[ (a_{\mu}^{hvp})_{\text{lat}} = (703.9 \pm 7.7) \cdot 10^{-10} \]

“No new physics”:  

\[ (a_{\mu}^{hvp})_{\text{NNP}} = (a_{\mu}^{hvp})_{\text{disp}} + (a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}}) \]
Lattice QCD approaches to HLbL

Matrix element of e.m. current between muon initial and final states:

\[
\langle \mu(p', s') | J_\mu(0) | \mu(p, s) \rangle = -e \bar{u}(p', s') \left( F_1(Q^2) \gamma_\mu + \frac{F_2(Q^2)}{2m} \sigma_{\mu\nu} Q_\nu \right) u(p, s)
\]

\[
a_\mu^{\text{HLbL}} = F_2(0)
\]

RBC/UKQCD:

* QCD + QED simulations

  [Hayakawa et al. 2005; Blum et al. 2015]

* QCD + stochastic QED

  [Blum et al. 2016, 2017]

Mainz group:

* Exact QED kernel in position space

  [Asmussen et al. 2015, 2016, and in prep.]

* Transition form factors of sub-processes

  [Gérardin, Meyer, Nyffeler 2016]

* Forward scattering amplitude

  [Green et al. 2015, 2017]
**Stochastic treatment of QED contribution:**

\[ G_{\mu\nu}(x, y) = \frac{1}{VT} \delta_{\mu\nu} \sum_{k, |k| \neq 0} \frac{e^{ik \cdot (x-y)}}{\hat{k}^2} \]

\[ (a_{h^hbl}^{l})_{con} = (116.0 \pm 9.6) \cdot 10^{-11} \]

\[ (a_{h^hbl}^{l})_{disc} = (-62.5 \pm 8.0) \cdot 10^{-11} \]

**Connected contribution:**

**Leading disconnected contribution:**

**Compute sub-leading disconnected diagrams**

**Study general systematic effects**

[Blum et al., Phys Rev D93 (2016) 014503]

Exact QED kernel in position space

- Determine QED part perturbatively in the continuum in infinite volume
  ⇒ no power-law volume effects

\[ a_\mu^{\text{hlbl}} = F_2(0) = \frac{m e^6}{3} \int d^4 y \int d^4 x \, \overline{\mathcal{L}}_{[\rho, \sigma]; \mu \nu \lambda}(x, y) i\Pi_{\rho; \mu \nu \lambda \sigma}(x, y) \]

- QCD four-point function:
  \[ i\Pi_{\rho; \mu \nu \lambda \sigma}(x, y) = - \int d^4 z \, z_\rho \left\langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0) \right\rangle \]

- QED kernel function: \( \overline{\mathcal{L}}_{[\rho, \sigma]; \mu \nu \lambda}(x, y) \) [Asmussen, Green, Meyer, Nyffeler, in prep.]
  - Infra-red finite; can be computed semi-analytically
  - Admits a tensor decomposition in terms of six weight functions which depend on \( x^2, y^2, x \cdot y \)
  ⇒ 3D integration instead of \( \int d^4 x \int d^4 y \)

- Weight functions computed and stored on disk
Preliminary results

* Accumulated connected contribution

\[
a^{chlbl}_\mu = \frac{m^6e}{3} 2\pi^2 \int_0^{y_{\text{max}}} |y|^3 \, dy \int d^4x \widetilde{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y) i\Pi_{\rho;\mu\nu\lambda\sigma}(x, y)
\]

Finite size effects (\(a = 0.086\) fm, \(m_\pi = 280\) MeV)

\begin{figure}
\centering
\includegraphics[width=\textwidth]{preliminary_results.jpg}
\caption{Accumulated connected contribution.}
\end{figure}

[Asmussen, Gérardin, Nyffeler, Meyer, arXiv:1811.08320]
Preliminary results

* Accumulated connected contribution

\[
a_\mu^{\text{chbl}} = \frac{me^6}{3} 2\pi^2 \int_0^{y_{\text{max}}} |y|^3 d|y| \int d^4x \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y) i\Pi_{\rho;\mu\nu\lambda\sigma}(x, y)
\]

Discretisation effects \((m_\pi = 280\text{ MeV})\)

* Controlled discretisation and finite-volume effects

[Asmussen, Gérardin, Nyffeler, Meyer, arXiv:1811.08320]
Transition form factor $\pi^0 \rightarrow \gamma^* \gamma^*$

* Pseudoscalar meson exchange expected to dominate LbL scattering  
[Nyffeler, arXiv:1602.03737]

* Compute transition form factor between $\pi^0$ and two off-shell photons:

$$\epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{\pi^0\gamma^*\gamma^*}(m_\pi^2; q_1^2, q_2^2) \equiv M_{\mu\nu}$$

$$M_{\mu\nu} \sim C_{\mu\nu}^{(3)}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) =$$

$$\sum_{\vec{x}, \vec{z}} \left\langle T \{ J_\nu(\vec{0}, \tau + t_\pi) J_\mu(\vec{z}, t_\pi) P(\vec{x}, 0) \} \right\rangle e^{i\vec{p} \cdot \vec{x}} e^{-i\vec{q}_1 \cdot \vec{z}}$$

* Kinematics:  
$$\vec{p} = 0, \quad q_1^2 = \omega_1^2 - |\vec{q}_1|^2, \quad q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$
Transition form factor $\pi^0 \rightarrow \gamma^*\gamma^*$

* Fit $Q^2$-dependence of $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ to several model ansätze; extrapolate to the physical point

* Results for $\pi^0$ contribution to hadronic light-by-light scattering:

$$ (a_{\mu}^{hbl})_{\pi^0} = \begin{cases} (59.7 \pm 3.6) \cdot 10^{-11} \\ (62.6^{+3.0}_{-2.5}) \cdot 10^{-11} \end{cases} $$

- Lattice QCD [Gérardin et al., arXiv:1903.09471]
- Dispersion theory [Hoferichter et al., arXiv:1903.09471]
**Total hadronic contribution**

- Combine lattice estimates for the HVP and HLbL contributions — very preliminary!
Muon anomalous magnetic moment

- One of the most promising hints for new physics
- Beautiful interplay between theory and experiment
- Numerous technical and computational challenges
- New experiments will significantly increase sensitivity
- Theory must keep pace

Lattice QCD

- Provides model-independent estimates for hadronic contributions
- HVP: difficult to reach sub-percent precision
- HLbL: 10–15% calculation will have great impact