Practice Exam
Theoretical Physics 3 : QM WS2020/2021
Lecturer : Prof. M. Vanderhaeghen

12.02.2021

Exercise 1. General questions. (20 points + 10 bonus)

1.1. *(5 p.) Momentum space.*

Consider the ground state wave function of the harmonic oscillator in spatial representation

$$\langle x|\psi_0\rangle = A_0\, e^{-\frac{m\omega}{2\hbar}x^2}.$$

Recall

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px} \quad \text{and} \quad \int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}.$$

Compute $\langle p|\psi_0\rangle$.

1.2. *(5 p.) Translation operator.*

Consider an operator $\hat{T}(a) \equiv e^{\frac{ia}{\hbar}\hat{p}}$, where $\hat{p}$ is the momentum operator and a is a real parameter.

a) Is it an observable? Why?

b) Show that $\hat{T}(a)\,\psi(x) = \psi(x+a)$.

1.3. *(5 p.) Time evolution operator.*

Assume $\hat{H}$ is the *time-independent* Hamiltonian.

a) Show that the operator $\hat{U}(t-t_0) \equiv e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$ is unitary.

b) Show that the solution to the time-dependent Schrödinger equation is

$$\Psi(x,t) = \hat{U}(t-t_0)\Psi(x,t_0),$$

with $\Psi(x,t_0)$ being a given wave function of the system at time t_0.

1.4. *(5 p.) Measurements.*

Consider two observables $\hat{A}$ and $\hat{B}$.
$\hat{A}$ has two normalized eigenstates $|a_1\rangle$ and $|a_2\rangle$, with eigenvalues a_1 and a_2, respectively.
$\hat{B}$ has two normalized eigenstates $|b_1\rangle$ and $|b_2\rangle$, with eigenvalues b_1 and b_2, respectively.
Assume the eigenstates are related by

$$|a_1\rangle = \frac{3}{5}|b_1\rangle + \frac{4}{5}|b_2\rangle \qquad |a_2\rangle = \frac{4}{5}|b_1\rangle - \frac{3}{5}|b_2\rangle\,.$$

a) The observable $\hat{A}$ is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?

b) If afterwards $\hat{B}$ is measured, what are the possible outcomes, and what are their probabilities?

c) Right after $\hat{B}$ is measured, $\hat{A}$ is measured again. What is the probability of getting a_1?

1.5. *(Bonus 10 p.) Eigenfunctions and degeneracy.*

a) *(2 p.)* What is the degree of degeneracy for the energy of a one-dimensional free particle?

b) *(3 p.)* Is the ground state of an infinite square well an eigenfunction of momentum? If so, what is its momentum? If not, why not?

c) *(5 p.)* Using the Schrödinger equation, prove that in one dimension there are no degenerate bound states.

Exercise 2. Half-harmonic oscillator. (25 points + 5 bonus)

Consider a particle of mass m, which is moving in one dimension in a "half"-harmonic potential $V(x)$

$$V(x) = \begin{cases} \infty, & x < 0; \\ \frac{1}{2}m\omega^2 x^2, & x \geq 0. \end{cases}$$

a) *(5 p.)* Write down the *stationary* Schrödinger equation for $x \geq 0$ using the dimensionless quantities

$$y = \sqrt{\frac{m\omega}{\hbar}}x \quad \text{and} \quad \varepsilon = \frac{E}{\hbar\omega}.$$

b) *(5 p.)* Show that the asymptotic behavior of the solution for large y is given by $e^{-y^2/2}$.

c) *(7 p.)* By separating the asymptotic behavior for $y \to \infty$, we define

$$\psi(y) = h(y)e^{-y^2/2}$$

Derive the equation for $h(y)$ for $y \geq 0$.

d) *(8 p.)* We know that for the *regular* quantum harmonic oscillator the eigenfunctions of the Hamiltonian are expressed in terms of the Hermite polynomials:

$$\psi_n(y) \propto H_n(y)e^{-y^2/2}, \quad n = 0, 1, 2, \dots ,$$

where the Hermite polynomials $H_n(y)$ satisfy the differential equation

$$H_n''(y) - 2yH_n'(y) + 2nH_n(y) = 0, \quad n = 0, 1, 2, \dots ,$$

and can equivalently be defined as

$$H_n(y) = (-1)^n e^{y^2} \frac{\partial^n}{\partial y^n} e^{-y^2}.$$

Deduce the spectrum in the case of the given *"half"*-harmonic potential.

e) *(Bonus 5 p.)* The Hermite polynomials are normalised as

$$\int_{-\infty}^{\infty} dy\, H_n(y)H_m(y)e^{-y^2} = 2^n n! \sqrt{\pi}\delta_{nm}.$$

What are the normalised *ground* state and *first excited* state wave functions of the given *"half"*-harmonic potential?

Exercise 3. Stark effect. (25 points)

In this problem we consider the modification (using first order perturbation theory) of the energy spectrum of the hydrogen atom placed in a static electric field.

Consider an electron in the $n = 2$ state of the hydrogen atom. The electric dipole moment $\vec{d} = -e\vec{r}$ of the electron interacts with an external electric field $\vec{E}$ through

$$\hat{H}'_E = -\vec{d} \cdot \vec{E},$$

which can be treated as a perturbation to the Coulomb potential.

Assume a constant electric field along the x-axis:

$$\vec{E} = E_0 \, \vec{e}_x.$$

We denote the unperturbed eigenstates $|n \, l \, m_l\rangle$ (neglecting spin) as

$$\begin{aligned} |1\rangle &\equiv |2\,0\,0\rangle, \\ |2\rangle &\equiv |2\,1\,0\rangle, \\ |3\rangle &\equiv |2\,1\,+1\rangle, \\ |4\rangle &\equiv |2\,1\,-1\rangle. \end{aligned}$$

a) *(15 p.)* Recall the hydrogen atom wave functions are given by

$$\psi_{nlm_l}(r, \theta, \phi) = R_{n,l}(r) \; Y_{l,m_l}(\theta, \phi).$$

You are given the spherical harmonics

$$\begin{aligned} Y_{0,0}(\theta, \phi) &= \frac{1}{\sqrt{4\pi}}, \\ Y_{1,0}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\ Y_{1,\pm 1}(\theta, \phi) &= \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}, \end{aligned}$$

and the radial integral

$$\int_0^\infty dr \, r^3 R_{2,0}(r) R_{2,1}(r) = 3\sqrt{3} \, a,$$

with a being the Bohr radius.

Determine the 4×4 matrix form of $\hat{H}'_E$ in the unperturbed basis in terms of $\Omega_e \equiv eE_0 \frac{a}{\hbar}$.

Hint: Use symmetry relations to argue that several of the angular integrals are zero.

b) *(10 p.)* Diagonalize the above matrix to calculate the first order corrections to all four $n = 2$ levels due to $\hat{H}'_E$ (you only need to find the eigenvalues, not the eigenstates).

Make a qualitative sketch of the total energy of the $n = 2$ levels as a function of the externally applied electric field E_0. Comment on their degeneracies.

Exercise 4. Spin-1/2 in a time dependent magnetic field. (30 points)

The neutron is a spin- particle. Its magnetic moment $\vec{\mu}_n$ is expressed in terms of its spin as $\vec{\mu}_n = \gamma_n \frac{\hbar}{2} \vec{\sigma}$, with $\gamma_n < 0$ being the neutron gyromagnetic ratio and $\vec{\sigma}$ being the vector of Pauli matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The magnetic moment couples to an external magnetic field $\vec{B}$ through the (interaction) Hamiltonian:

$$\hat{H} = -\vec{\mu}_n \cdot \vec{B}.$$

Consider a uniform external magnetic field $\vec{B}$, which has a constant component along the z-axis, and a rotating component in the xy-plane

$$\vec{B} = B_1 \, \vec{e}_x \, \cos\omega(t_0 + t) \, + \, B_1 \, \vec{e}_y \, \sin\omega(t_0 + t) \, + \, B_0 \, \vec{e}_z,$$

where B_0 and B_1 are constant amplitudes and ω is the (externally controlled) frequency.

a) *(5 p.)* Write down $\hat{H}$ in 2×2 matrix form using $\omega_0 \equiv -\gamma_n B_0$ and $\omega_1 \equiv -\gamma_n B_1$.

b) *(5 p.)* The neutron spin- state at time t is given (in matrix notation) by

$$\Psi(t) = \begin{bmatrix} c_+(t) \\ c_-(t) \end{bmatrix},$$

where $c_+(t)$ and $c_-(t)$ are the amplitudes of being in spin-up and spin-down states, respectively. Given the time-dependent Schrödinger equation

$$\hat{H}\Psi(t) = i\hbar \frac{\partial}{\partial t}\Psi(t),$$

write down the system of equations which describes the time evolution of $c_\pm(t)$.

c) *(5 p.)* Consider the special case of the resonance condition $\omega = \omega_0$. Express

$$\begin{aligned} c_+(t) &= e^{-\frac{i}{2}\omega_0 t} \, \beta_+(t), \\ c_-(t) &= e^{+\frac{i}{2}\omega_0 t} \, \beta_-(t), \end{aligned}$$

and write down the equivalent differential equations for $\beta_\pm(t)$.

d) *(10 p.)* Given the values $c_+(0)$ and $c_-(0)$ at time $t = 0$, show that the general solution for $c_+(t)$ is

$$c_+(t) \; = \; e^{-i\chi} \cos\phi \;\, c_+(0) \;\; - \;\; ie^{-i\delta} \sin\phi \;\, c_-(0) \;\; ,$$

with $\phi \equiv \frac{\omega_1}{2}t$, $\chi \equiv \frac{\omega_0}{2}t$ and $\delta \equiv \frac{\omega_0}{2}(t + 2t_0)$.

e) *(5 p.)* Using the general solution for $c_\pm(t)$ which is then given by

$$\begin{bmatrix} c_+(t) \\ c_-(t) \end{bmatrix} = \begin{bmatrix} e^{-i\chi}\cos\phi & -ie^{-i\delta}\sin\phi \\ -ie^{+i\delta}\sin\phi & e^{+i\chi}\cos\phi \end{bmatrix} \begin{bmatrix} c_+(0) \\ c_-(0) \end{bmatrix},$$

determine the probabilities to find the neutron in the spin-up and spin-down states at time t. Sketch the probabilities for $c_-(0) = 0$.