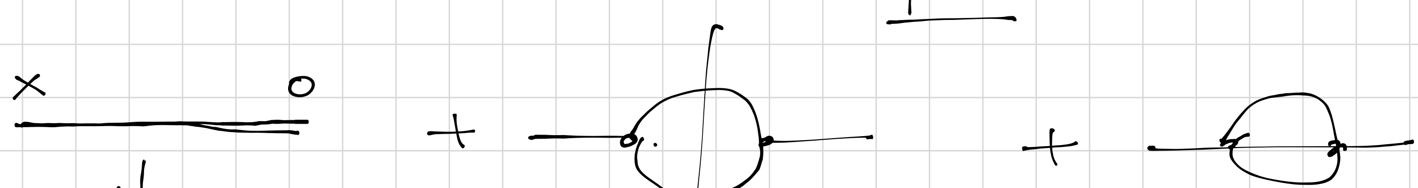
Lecture 26

Källen–Lehmann representation

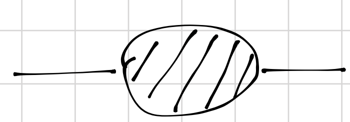
$$\int d^4x\, e^{ipx} \langle \Omega | T(\phi(x)\,\phi(0)) | \Omega \rangle = \frac{1}{2\pi}\int_0^\infty dM^2\, \rho(M^2)\, \frac{i}{p^2 - M^2 + i\varepsilon}$$

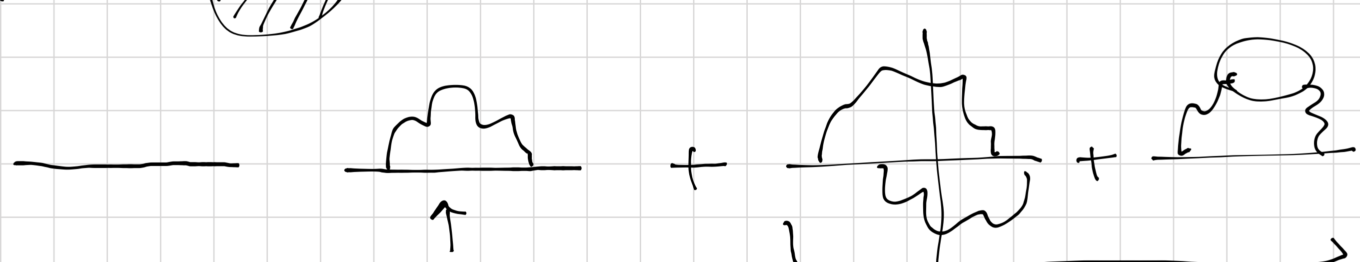
$$\sum_\lambda \int \frac{d^3\vec{q}}{(2\pi)^3}\, \frac{1}{2E_q^\lambda}\, |\lambda_q\rangle\langle\lambda_q|$$

$$|\lambda_q\rangle = U_q |\lambda_0\rangle \qquad \vec{q} = 0$$



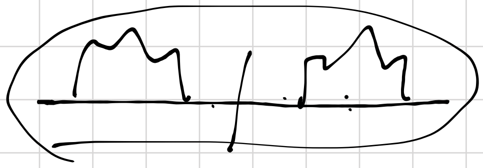
$$\delta(p^2 - m_\lambda^2) \qquad \sqrt{1 - \frac{4m^2}{p^2}}\;\theta(p^2 - 4m^2)$$

$i\Sigma(\not p) =$ 

$=$ 

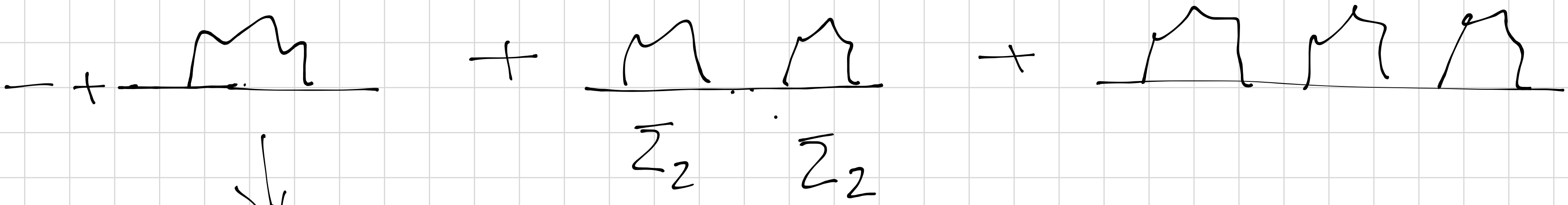
Σ_2 $\qquad$ Σ_4

1-particle irreducible

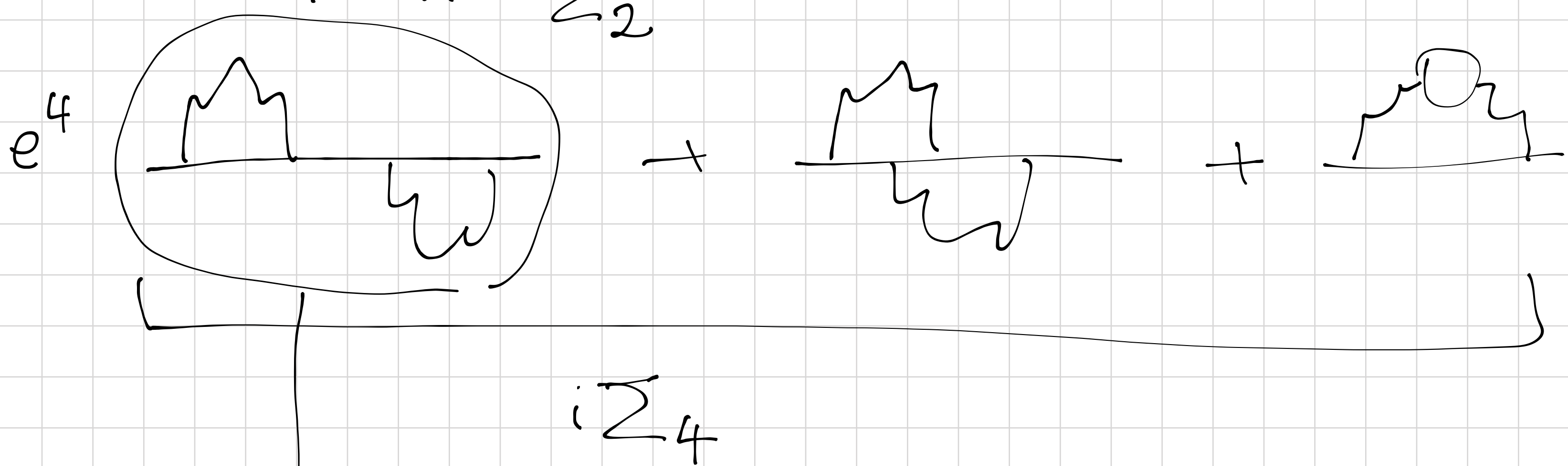
e^4 

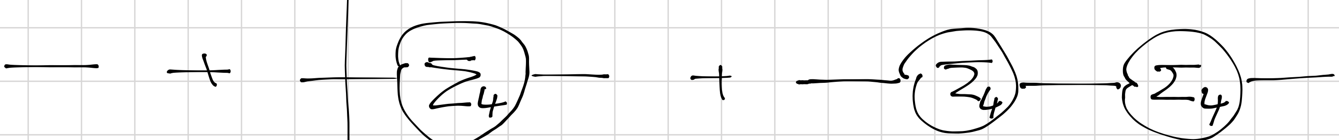
$$\frac{i}{\not p - m}\; i\Sigma_2\; \frac{i}{\not p - m} + \dots$$

We want to resum to all orders



$$\frac{1}{p^2 - M^2 - \Sigma_2}$$



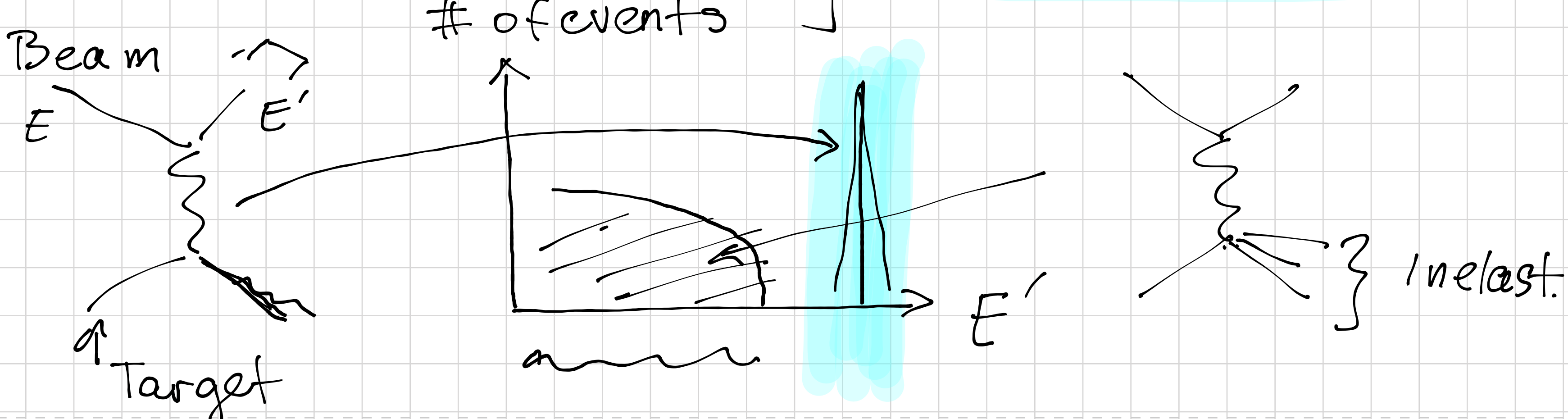


has to be excluded: makes part of the resummed Σ_2

$$f(x) = \sum c_n x^n = \sum c_{1n} x^n + \sum c_{2n} (x^2)^n + \ldots$$

Asymptotic states are single particles

$$\prod_{j=1}^{n} \left[i \int d^4 x_j \, e^{\pm i p_j x_j} (\Box + m^2) \right] \langle \Omega | T(\phi(x_1) \ldots \phi(x_n)) | \Omega \rangle$$

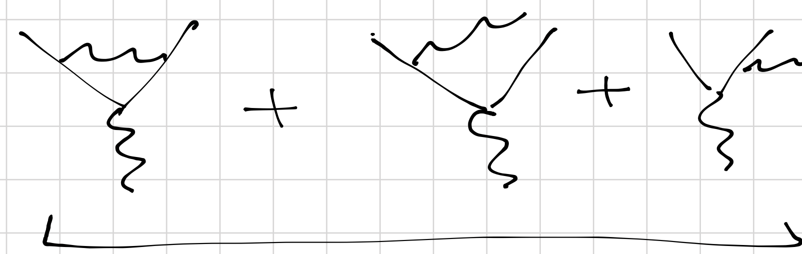


$$\Box + m^2 = -p^2 + m^2$$

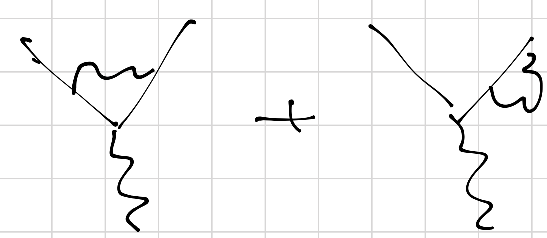
$$\langle 0 | T(\phi(x)\phi(0)) | 0 \rangle \stackrel{\sim}{=} \frac{i}{p^2 - m^2 + i\varepsilon}$$

$$\langle \Omega | T(\phi(x)\phi(0)) | \Omega \rangle = \text{pole} + \text{cut}$$

We observed some ∞ cancellations

1. IR $\longrightarrow$ combined 

$\ln(\lambda) \longrightarrow \varnothing$

2. UV $\longrightarrow$ combined 

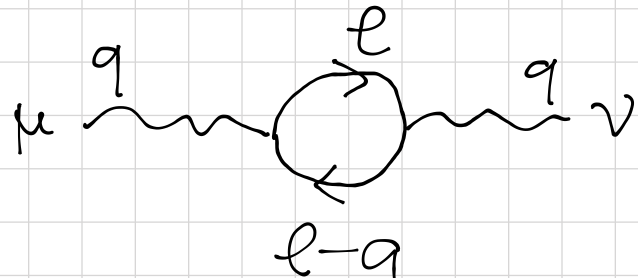
$\ln(\Lambda^2) \longrightarrow \varnothing$

$$\delta F_1(0) + \left.\frac{d\Sigma(\not p)}{d\not p}\right|_{\not p = m} = 0$$

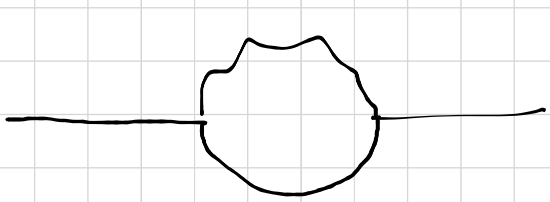
If something is exactly $\varnothing$ $\longrightarrow$ symmetry

Gauge invariance $\longrightarrow$ Ward identities

Today: renormalization of e. charge



$$\frac{-ig^{\mu\nu}}{q^2 + i\varepsilon} \longrightarrow \frac{-ig^{\mu\nu}}{q^2 - \underbrace{\Sigma(q)}_{\text{could it generate } m_\gamma?}}$$



$$\frac{i}{\not p - m} \longrightarrow \frac{i}{\not p - m - \Sigma(\not p)}$$

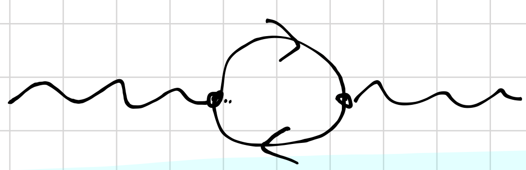
→ Recall that massive spin-1 theory is not gauge invariant

$$\mathcal{L} = -\frac{1}{4}(F^{\mu\nu})^2 - \underbrace{\frac{1}{2}m^2 A^2}_{\rightarrow \text{ breaks g.i.}}$$

$$\langle \Omega | T(A^\mu(x) A^\nu(y)) | \Omega \rangle = \int \frac{d^4 q}{(2\pi)^4} e^{iqx} \, i\, G^{\mu\nu}(q)$$

$$i G^{\mu\nu}(q) = \frac{-i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}{q^2 + i\varepsilon} + O(e^2)$$

$$i G^{\mu\nu}(q) = A(q^2) g^{\mu\nu} + B(q^2) q^\mu q^\nu$$

$$i \Pi^{\mu\nu}(q) =$$ 

$$= (-)(-ie)^2 \int \frac{d^4 \ell}{(2\pi)^4} \frac{i}{(\ell^2 - m^2 + i\varepsilon)} \frac{i}{((\ell-q)^2 - m^2 + i\varepsilon)}$$
$$\mathrm{Tr}\left[\gamma^\nu (\not{\ell} - \not{q} + m)\gamma^\mu (\not{\ell} - m)\right]$$

Check Ward ID first!

$$q_\mu \Pi^{\mu\nu} = 0 \quad \text{(if gauge-inv.)}$$

$$\mathrm{Tr}\left[\gamma^\nu (\not{\ell} - \not{q} + m) \not{q} (\not{\ell} + m)\right]$$

$$\not{q} = (\not{\ell} - m) - (\not{\ell} - \not{q} - m)$$

$$= (\ell^2 - m^2)\, 4 (\ell - q)^\nu - [(\ell-q)^2 - m^2) \cdot 4 \ell^\nu$$

$$\int \frac{d^4\ell}{(2\pi)^4}\left[\frac{4(\ell-q)^\nu}{(\ell-q)^2-m^2} - \frac{4\ell^\nu}{\ell^2-m^2}\right] = 0$$

$$q_\mu \Pi^{\mu\nu} = q_\nu \Pi^{\mu\nu} = 0$$

$$\hookrightarrow \Pi^{\mu\nu}(q) = \Pi(q^2)\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]$$

$$\mathrm{Tr}[\ldots] = 4\left[(\ell-q)^\mu \ell^\nu + (\ell-q)^\nu \ell^\mu + (m^2 - \ell(\ell-q))g^{\mu\nu}\right]$$

$$\frac{1}{(\ell^2-m^2)((\ell-q)^2-m^2)} = \int_0^1 \frac{dx}{\left[(\ell-xq)^2 - \underbrace{\left[m^2 - x(1-x)q^2\right]}_{\Delta} + i\varepsilon\right]^2}$$

First warning:

$$\int d^4\ell \; \frac{[\ell^2, \ell; \ell^0]}{\ell^4} \qquad \textit{divergent}$$

Shift $\int$ momentum from ℓ to $\ell - xq$

$$\int d^4\tilde{\ell} \;\ldots (\tilde{\ell} + xq)$$ suppress tilde

$$\mathrm{Tr}[\ldots] = 4\Big[(\ell-(1-x)q)^\mu(\ell+xq)^\nu + (\ell-(1-x)q)^\nu(\ell+xq)^\mu + (m^2 - (\ell-(1-x)q, \ell+xq))g^{\mu\nu}\Big]$$

$$-4e^2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{2\ell^\mu\ell^\nu - \ell^2 g^{\mu\nu} - 2x(1-x)q^\mu q^\nu + (m^2 + x(1-x)q^2)g^{\mu\nu}}{\left[\ell^2 - \Delta + i\varepsilon\right]^2}$$

$$= i\Pi^{\mu\nu} = i\Pi(q^2)\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]$$

Numerator : $\underline{2x(1-x)\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]}$

$$+ g^{\mu\nu}\underbrace{\left[m^2 - x(1-x)q^2 - \frac{1}{2}\ell^2\right]}_{\text{has to give } \emptyset}$$

$\ell^\mu \ell^\nu = \frac{1}{4} g^{\mu\nu}\ell^2$

$\Delta - \frac{1}{2}\ell^2$

" $1 + 2 + 3 + 4 + \ldots + \infty = -1/12$ "

$$I(m,n) \equiv \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell^2)^m}{\left[\ell^2 - \Delta + i\varepsilon\right]^n}$$

$$= + \frac{i}{(4\pi)^2}(-1)^{m-n} \frac{\Gamma(n-m-2)\,\Gamma(m+2)}{\Gamma(n)} \Delta^{-n+m+2}$$

$$\Delta \cdot I(0,n) - \frac{1}{2} I(1,n) \Big|_{n=2}$$

$\Gamma(n) = (n-1)!$

$$\frac{i}{(4\pi)^2}(-1)^n \left[\Delta \cdot \frac{\Gamma(n-2)\,\cancel{\Gamma(2)}^{\,1}}{\Gamma(n)} \Delta^{-n+2} + \cancel{\frac{1}{2}} \frac{\Gamma(n-3)\,\cancel{\Gamma(3)}}{\Gamma(n)} \Delta^{-n+3}\right]$$

$$= \frac{i}{(4\pi)^2}(-1)^n \frac{\Delta^{-n+3}}{\Gamma(n)} \underbrace{\left[\Gamma(n-2) + \Gamma(n-3)\right]}_{\text{Use } \Gamma(x)\cdot x = \Gamma(x+1)}$$

$$\Gamma(n-2) = (n-3)\,\Gamma(n-3)$$

Now set $\underline{n=2}$

$$\frac{i}{(4\pi)^2}\,\frac{\Delta}{\Gamma(2)}\cdot\Gamma(-1)\,(n-2) = 0$$
$$-\infty \qquad 0$$

Instead of playing with powers m, n

$\longrightarrow$ Dimensional Regularization

play with $D = 4 - \varepsilon$

$\varepsilon \to$ small but finite

$$\Gamma\left(2-\frac{D}{2}\right) + \Gamma\left(1-\frac{D}{2}\right)$$
$$= \Gamma\left(\frac{\varepsilon}{2}\right) + \Gamma\left(-1+\frac{\varepsilon}{2}\right) = \Gamma\left(-1+\frac{\varepsilon}{2}\right)\cdot\left(\cancel{-1}+\frac{\varepsilon}{2}+\cancel{1}\right)$$

Dim Reg is trickier than this

$\longrightarrow$ $\mathrm{Tr}\,\gamma^\mu\gamma^\nu = D$; $\gamma^\mu\gamma^\alpha\gamma_\mu = (2-D)\gamma^\alpha$ etc.

$$i\Pi^{\mu\nu} = -8e^2\int_0^1 dx\, x(1-x)\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]$$

$$\int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\left[\ell^2-\Delta+i\varepsilon\right]^2} \qquad \Delta = m^2 - x(1-x)q^2$$

$$\int_{\Lambda^2}^{m^2} dm^2\,\underbrace{\frac{\partial}{\partial m^2} I(0,2)} = \frac{2}{\Gamma(3)}\int_{\Lambda^2}^{m^2} dm^2\,\frac{-i}{(4\pi)^2}\,\frac{1}{m^2-x(1-x)q^2}$$

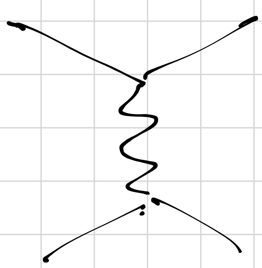
$$\wr\ I(0,3) = -\frac{i}{(4\pi)^2}\,\frac{\Gamma(1)\,\Gamma(2)}{\Gamma(3)}\,\frac{1}{\Delta}$$

$$= -\frac{i}{(4\pi)^2}\,\ln\frac{m^2 - x(1-x)q^2}{\Lambda^2}$$

$$\Downarrow$$

$$\Pi^{\mu\nu} = -i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\frac{8e^2}{(4\pi)^2}\int_0^1 dx\; x(1-x) \times \ln\frac{\Lambda^2}{m^2 + Q^2 x(1-x)}$$

$$\underline{Q^2 = -q^2}$$

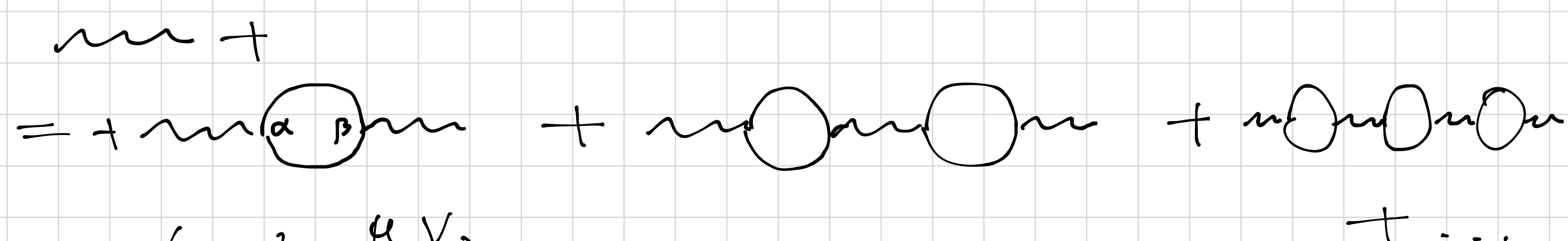


$$q^2 < 0 \qquad\qquad \alpha = \frac{e^2}{4\pi}$$

$$\Pi(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\,\ln\frac{\Lambda^2}{m^2 + Q^2 x(1-x)}$$

"Dressed" photon propagator

$i D^{\mu\nu} =$



$$= \frac{-i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}{q^2 + i\varepsilon} + \frac{-i\left(g^{\mu\alpha} - \frac{q^\mu q^\alpha}{q^2}\right)}{q^2 + i\varepsilon}\; i\Pi_{\alpha\beta}\,\frac{-i\left(g^{\nu\beta} - \frac{q^\nu q^\beta}{q^2}\right)}{q^2 + i\varepsilon}$$

$$+ \frac{-i\left(g^{\mu\alpha} - \frac{q^\mu q^\alpha}{q^2}\right)}{q^2 + i\varepsilon}\; i\Pi_{\alpha\beta}\,\frac{-i\left(g^{\beta\gamma} - \frac{q^\beta q^\gamma}{q^2}\right)}{q^2 + i\varepsilon}\; i\Pi_{\gamma\delta}\,\frac{-i\left(g^{\delta\nu} - \frac{q^\delta q^\nu}{q^2}\right)}{q^2 + i\varepsilon}$$

$$+ \ldots$$

$$\Pi_{\alpha\beta} \cdot \left(g^{\alpha\nu} - \frac{q^\alpha q^\nu}{q^2}\right) = \Pi(q^2)\left(q^2 g_{\alpha\beta} - q_\alpha q_\beta\right)\left(g^{\alpha\nu} - \frac{q^\alpha q^\nu}{q^2}\right)$$

$$= \Pi(q^2)\left(q^2 g_\beta{}^\nu - q_\beta q^\nu\right)$$

$$\frac{-i}{q^2}\left(g^{\mu\alpha} - \frac{q^\mu q^\alpha}{q^2}\right) \cdot \frac{\Pi(q^2)}{\cancel{q^2}}\left(\cancel{q^2} g_\alpha{}^\nu - \cancel{q_\alpha q^\nu}\right) = \frac{-i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}{q^2 + i\varepsilon} \times \Pi(q^2)$$

$$\Downarrow$$

$$iD^{\mu\nu} = \frac{-i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}{q^2 + i\varepsilon}\underbrace{\left[1 + \Pi + \Pi^2 + \dots\right]}_{= \frac{1}{1-\Pi}}$$

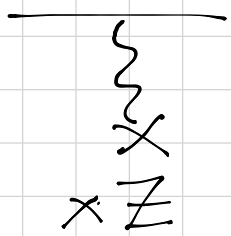
$$\Pi = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\, \ln \frac{\Lambda^2}{m^2 + Q^2 x(1-x)}$$

Measurement never gives you a bare propagator (e.g. Coulomb inter.)

$\longrightarrow$ only renormalized one

Coulomb potential at long distances

$$\sim -\frac{Z\alpha}{r} \longrightarrow -\frac{Z\alpha}{Q^2}$$



$$\Big|_{Q^2\to 0} \quad V_{eff.} = -\frac{Z\alpha \to \frac{e_0^2}{4\pi}}{Q^2\left(1-\Pi(0)\right)}$$

$$\Pi(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x) \ln\frac{\Lambda^2}{m^2}$$

$$= -\frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{m^2}$$

$$\Downarrow \quad e_{eff}^2 = \frac{e_0^2}{1-\frac{e_0^2}{12\pi^2}\ln\frac{\Lambda^2}{m^2}}$$

1. VP does not renormalize the photon mass
2. renormalizes electric charge (the strength of e.-m. interaction)

$$\alpha_{em} \simeq \frac{1}{137.03599} \equiv \frac{e_{eff}^2}{4\pi}$$

$$\alpha_{em}(Q^2) = \frac{\alpha_{em}}{1-\underbrace{\frac{\alpha}{3\pi}\ln\frac{Q^2}{m^2}}} \qquad Q^2 \gg m^2$$

Leading log approximation

Running coupling (constant)

$\alpha_{em}(Q^2)$ has a pole (Landau pole)

at $1-\frac{\alpha}{3\pi}\ln\frac{Q^2}{m^2}=0$ $\quad Q^2 = m^2 e^{\frac{3\pi}{\alpha}} \gg 10^{286}\,eV^2$

QED is not a complete theory
breaks down at (very) high energies

We renormalize α at some scale m^2
But we could've done so at a different scale $\mu^2 \longrightarrow \alpha_R(\mu^2)$

$$\downarrow$$

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha_R(\mu^2)} - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu^2}\right)$$

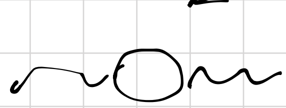
$$\frac{d}{d\mu^2}: \quad 0 = -\frac{1}{\alpha_R^2}\frac{d\alpha_R(\mu^2)}{d\mu^2} + \frac{1}{3\pi\mu^2}$$

$$\downarrow$$

$$\underbrace{\mu^2 \frac{d\alpha(\mu^2)}{d\mu^2} = \frac{\alpha^2}{3\pi}}_{\text{RGE}} \longleftarrow \underline{\beta\text{- function}}$$

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 + \beta_0\,\alpha(\mu^2)\ln\left(\frac{Q^2}{\mu^2}\right)}$$

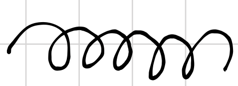
$$\beta(\alpha) = \underline{\beta_0}\,\alpha^2 + \underline{\beta_1}\,\alpha^3 + \ldots$$

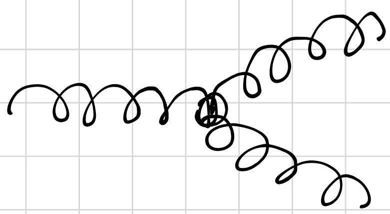
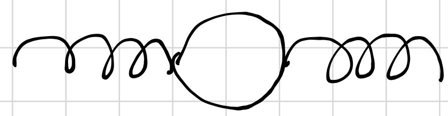
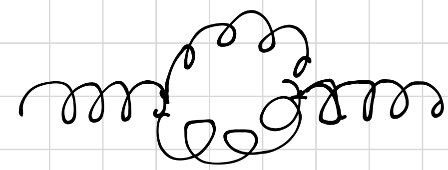



QED $\beta_0 = -\frac{1}{3\pi}$

QED is asymptotically strongly bound

QCD $\beta_0 > 0$ QCD is asymptoticaly free

 gluon propagator (massless) $-\dfrac{g^{\mu\nu}}{q^2}$

$$\beta_0 = \frac{-2n_f + 11 N_c}{12\pi}$$

N_c = # of colors (strong charge)

N_c in QED $= 1$

N_c in QCD $= 3$

n_f = # of flavors (types of fermions) $= 6$

Quark doublets $\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}$

$$\beta_0(N_c = 3,\ n_f = 6) = \frac{33-12}{12\pi} = \frac{7}{4\pi} > 0$$