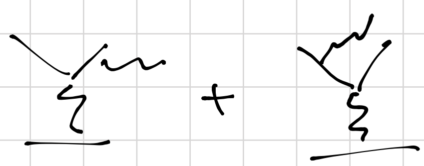
Lecture 25

Steps towards renormalization of QED

Last week: soft Bremsstrahlung (photon emission)

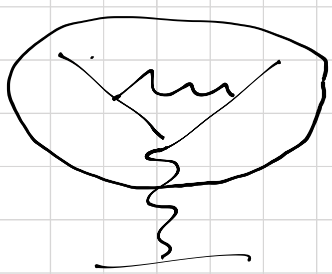


$\longrightarrow$ IR singularity due to $\varnothing$ photon mass

$\hookrightarrow \int_\lambda \frac{d\omega_\gamma}{\omega_\gamma} \sim -\ln\lambda$

elastic

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_0}{d\Omega} + \frac{d\sigma_{1\gamma}}{d\Omega}\bigg|_{\omega_\gamma < \underline{\Delta E}} \;\underset{\sim}{=}\; \frac{d\sigma_0}{d\Omega}\left(1 + \frac{\alpha}{\pi}\ln\frac{\Delta E^{2\gamma}}{\lambda^2}\left(\ln\frac{-t}{m^2} - 1\right)\right)$$



Vertex correction

UV and IR divergences

$$\Gamma^\mu = 1\cdot\gamma^\mu + 0\cdot i\sigma^{\mu\alpha}\frac{q_\alpha}{2m} \qquad \text{tree level}$$

$$= F_1(q^2)\gamma^\mu - \frac{i\sigma^{\mu\alpha}q_\alpha}{2m}F_2(q^2) \qquad \text{general case}$$

We evaluated the loop diag. (almost)

$\longrightarrow$ obtained a general expr. for a loop int.

$$\delta\Gamma^\mu = \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)$$

$$\cdot\left\{ -\frac{m\, i\sigma^{\mu\alpha}q_\alpha \cdot z(1-z)}{m^2(1-z)^2 + z\lambda^2 - xyq^2}\right.$$

$$\left. + \gamma^\mu\left[\ln\frac{z\Lambda^2}{m^2(1-z)^2 + z\lambda^2 - xyq^2} + \frac{m^2(1-4z+z^2) + (1-x)(1-y)q^2}{m^2(1-z)^2 + z\lambda^2 - xyq^2}\right]\right.$$

$$\delta F_2(0) = \frac{\alpha}{2\pi} \qquad \text{Schwinger in 1947}$$

anomalous magnetic moment

$\frac{\alpha}{2\pi} \simeq 1 \cdot 10^{-3}$

μ_e : exp vs. theory at 10^{-13}

μ_μ : 10^{-10}

We will discuss the renormalization in full detail : $F_1(0) \longleftrightarrow e_R$ from Coulomb strength

(recall ϕ^4 theory)

$$\lambda \to \lambda_R \equiv -\mathcal{M}(s_0)$$

$\delta F_1(q^2)$: $\underline{\delta F_1(0) = 0}$ $\longrightarrow$ e_R not renormalized

$$\hookrightarrow \delta F_1(0) = \frac{\alpha}{2\pi}\int_0^1 dz\,(1-z)$$

$$\left\{ \ln \frac{z\Lambda^2}{m^2(1-z)^2 + z\lambda^2} + \frac{m^2(1-4z^2+z^2)}{m^2(1-z)^2+z\lambda^2} \right\}$$

$$\delta F_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)$$

$$\left[\frac{m^2(1-4z+z^2) + (1-x)(1-y)q^2}{m^2(1-z)^2 + \underset{\sim}{z}\lambda^2 - xyq^2} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2+z\lambda^2} \right]$$

IR divergence

$z \to 1$

$m^2(1-z)^2 \to 0$

$x, y \to 0 \qquad \delta(1-z-x-y)$

$$\delta F_1(q^2) = -\frac{\alpha}{2\pi}\ln\frac{m^2}{\lambda^2}\int_0^1 dy \left[\frac{m^2 - q^2/2}{m^2 - q^2 y(1-y)} - 1 \right]$$

$$- \frac{\alpha}{2\pi}\int_0^1 dy\,\frac{m^2 - q^2/2}{m^2 - q^2 y(1-y)} \ln\left[1 - \frac{q^2}{m^2}y(1-y)\right]$$

$$\int_0^1 dy\, \frac{m^2 - q^2/2}{m^2 - q^2 y(1-y)} \Bigg|_{q^2 \gg m^2} \cong \ln\left(\frac{-q^2}{m^2}\right)$$

$$\delta F_1(q^2) = -\frac{\alpha}{2\pi} \ln\left(\frac{m^2}{\lambda^2}\right)\left[\ln\left(\frac{-q^2}{m^2}\right) - 1\right]$$

$$+ \text{ IR finite}$$

$$\sim \ln^2\left(\frac{-q^2}{m^2}\right)$$

Now can combine the IR div. at the level of cross section

$$\frac{d\sigma}{d\Omega}\bigg|_{O(\alpha)} = \frac{d\sigma_0}{d\Omega}\left(1 + \delta F_1(q^2)\right)^2$$

$$+ \frac{d\sigma_{1\gamma}}{d\Omega}$$

$$= \frac{d\sigma_0}{d\Omega}\left[1 \underline{- 2\cdot\frac{\alpha}{2\pi} \ln\frac{m^2}{\lambda^2}\left(\ln\frac{-q^2}{m^2} - 1\right)}\right]$$

$$+ \frac{d\sigma_0}{d\Omega} \cdot \underline{\frac{\alpha}{\pi} \ln\left(\frac{\Delta E^2}{\lambda^2}\right)\left(\ln\frac{-q^2}{m^2} - 1\right)}$$

<u>IR div. cancel exactly order by order</u>

What happens to $\delta F_1(0)$ and respective IR and UV divergences?

<u>Field-strength renormalization</u>

$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$

$H = H_0 + H_{int.}$

$|\Omega\rangle \longrightarrow$ true vacuum of interacting th.

$|0\rangle \longrightarrow$ vacuum of free theory H_0

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = D_F(x-y; m^2) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon}$$

p-space : $\int d^4x \, e^{ipx} \langle 0 | T \phi(x) \phi(0) | 0 \rangle = \frac{i}{p^2 - m^2 + i\varepsilon}$

What happens to it in interacting theory?

Insert a complete set of states between $\phi(x)$ and $\phi(y)$

$$\mathbb{1}_{1-part.} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} |p\rangle\langle p|$$

Hilbert space of $(H_0, \vec{P})$

Interacting theory $\longrightarrow$ Fock space (not just 1-part.)

$|\lambda_p\rangle$: $\vec{P} |\lambda_p\rangle = \vec{p} |\lambda_p\rangle$

$|\lambda_p\rangle = U(p) |\lambda_0\rangle$: $E_p^\lambda = \sqrt{\vec{p}^2 + m_\lambda^2}$

1-part, 2-part, ... ∞

$$\mathbb{1} = \underbrace{|\Omega\rangle\langle\Omega|}_{} + \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p^\lambda} |\lambda_p\rangle\langle\lambda_p|$$

$\langle \Omega | \phi(x) | \Omega \rangle \langle \Omega | \phi(y) | \Omega \rangle \longrightarrow 0$

$x^0 > y^0$

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p^\lambda} \langle \Omega | \phi(x) | \lambda_p \rangle \langle \lambda_p | \phi(y) | \Omega \rangle$$

$$\langle \Omega | \phi(x) | \lambda_p \rangle = \langle \Omega | \phi(0) | \lambda_p \rangle e^{-ipx}$$

$$\phi(x) = e^{iPx} \phi(0) e^{-iPx}$$

$$e^{-iPx} | \Omega \rangle = | \Omega \rangle$$

$$P | \Omega \rangle = 0$$

$$e^{-iPx} | \lambda_p \rangle = e^{-ipx} | \lambda_p \rangle \Big|_{p^0 = E_p^\lambda}$$

$$U^{-1}(p) | \lambda_p \rangle = | \lambda_0 \rangle$$

$$\underbrace{\langle \Omega | U}_{\langle \Omega |} \underbrace{U^{-1} \phi(0) U}_{= \phi(0)} \underbrace{U^{-1} | \lambda_p \rangle}_{| \lambda_0 \rangle} = \langle \Omega | \phi(0) | \lambda_0 \rangle$$

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4} \frac{i \cdot e^{-ip(x-y)}}{p^2 - m_\lambda^2 + i\varepsilon} \cdot \left| \langle \Omega | \phi(0) | \lambda_0 \rangle \right|^2$$

Källen – Lehmann representation

$$\langle \Omega | T \underline{\phi(x) \phi(y)} | \Omega \rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \underline{D_F(x-y; M^2)}$$

$$\rho(M^2) = \sum_\lambda 2\pi \delta(M^2 - m_\lambda^2) \left| \langle \Omega | \phi(0) | \lambda_0 \rangle \right|^2$$

$\rho \longrightarrow$ particle content of th.; $D_F \longrightarrow$ spin of ϕ

Analytical structure of the spectral fn.

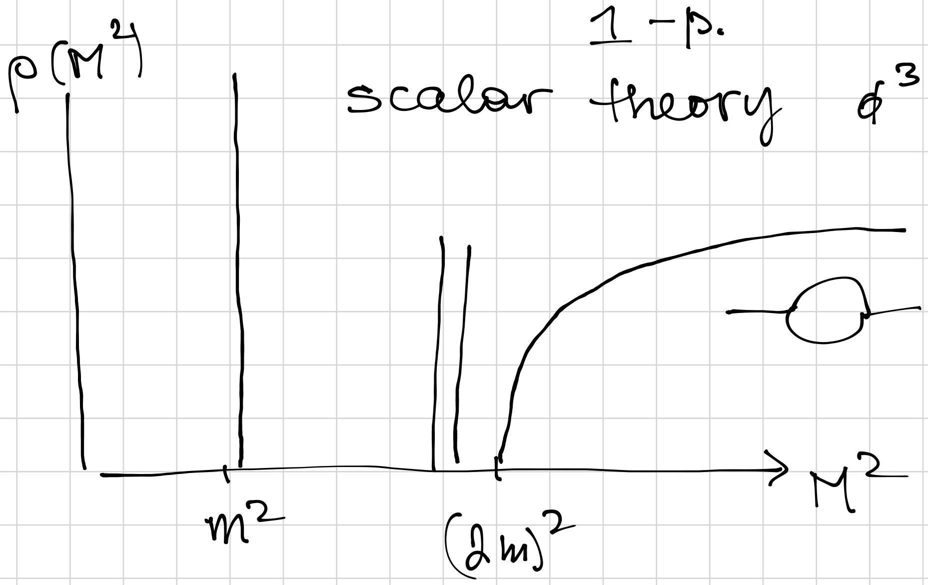
1 - par. state $\longrightarrow$ $\delta(M^2 - m_1^2)$

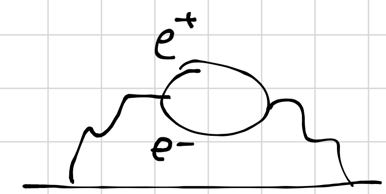
2 - part. $\longrightarrow$ continuum

$\vdots$

There could be bound states

1 - p. 2 - p. 3 p





$e^+e^- \rightarrow$ bound st positronium

$$\rho(M^2) = 2\pi\,\delta(M^2 - m^2)\cdot Z + \text{multi-part. cont.}$$

$$Z = \left|\langle \Omega | \phi(0) | \lambda_0^{1-p.} \rangle\right|^2$$

probability for ϕ to create a 1 - particle state from Ω

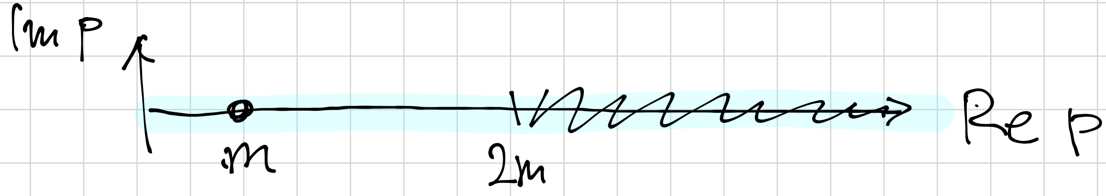
The mass $m \longrightarrow$ physical mass $\leftrightarrow$ observable

$m_0 \longrightarrow$ bare mass that enters $\mathcal{L}$

$m \neq m_0$

$$\int d^4x\, e^{ipx} \langle \Omega | T\phi(x)\phi(y) | \Omega \rangle$$

$$= \frac{iZ}{p^2 - m^2 + i\varepsilon} + \int_{4m^2}^{\infty} \frac{dM^2}{2\pi}\, \rho(M^2)\, \frac{i}{p^2 - M^2 + i\varepsilon}$$



$$\int d^4x\, e^{ipx} \langle 0 | T(\ \ ) | 0 \rangle = \frac{i}{p^2 - m^2 + i\varepsilon}$$

$\langle 0 | \phi(0) | p \rangle$ $\qquad |p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle$

$$\langle 0 | \phi(0) | p \rangle = \langle 0 | \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} (a_k + \cancel{a_k^\dagger}) \sqrt{2E_p}\, a_p^\dagger |0\rangle$$

$$[a_k, a_p^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{k}) \implies \langle 0 | 0 \rangle = \underline{1}$$

Interacting theory: $Z \neq 1$

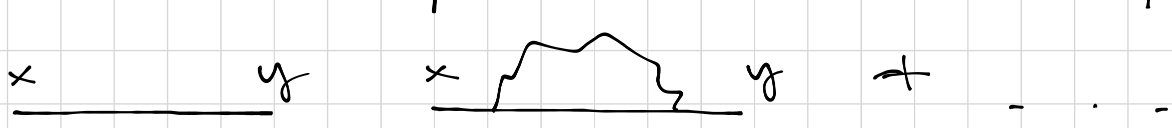
Generalize to spin $\neq 0$

$$\langle \Omega | T \psi(x) \bar{\psi}(0) | \Omega \rangle = \frac{i Z_2 \sum_s u_s(p) \bar{u}_s(p)}{p^2 - m^2 + i\varepsilon} \dots$$

$$= Z_2 \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} \dots$$

$$\langle \Omega | \psi(0) | p, s \rangle = \sqrt{Z_2}\, u_s(\underline{p})$$

Consider perturbative expansion in QED



0^{th} order $\qquad \dfrac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\varepsilon}$

2-part.

1^{st} order (in α) $\qquad \dfrac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\varepsilon} \left[-i \Sigma_2(p) \right] \dfrac{i(\not{p} + m_0)}{p^2 - m_0^2 + i}$

$$-i \Sigma_2(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not{k} + m_0) \gamma_\mu}{[k^2 - m_0^2 + i\varepsilon][(p-k)^2 - \lambda^2 + i\varepsilon]}$$

We can calculate this diagram

$$\frac{1}{k^2-m_0^2}\;\frac{1}{(p-k)^2-\lambda^2} \longrightarrow \text{combine w. Feynman p.}$$

$$\hookrightarrow \quad -i\Sigma_2(p) = -e^2\int_0^1 dy \int \frac{d^4\ell}{(2\pi)^4} \frac{(-2y\not{p}+4m_0)}{[\ell^2-\Delta+i\varepsilon]^2}$$

$\Delta = (1-y)m^2 - y(1-y)p^2 + y\lambda^2$ divergent.

To still evaluate:

$$\frac{\partial}{\partial\lambda^2}\frac{1}{[\ell^2-\Delta+i\varepsilon]^2} = \underbrace{\frac{2y}{[\ell^2-\Delta+i\varepsilon]^3}}$$

$$\Sigma_2(p) = \int_{\Lambda^2}^{\lambda^2} d\lambda^2 \left[\frac{\partial}{\partial\lambda^2}\Sigma_2(p)\right] = \frac{\alpha}{2\pi}\int_0^1 dy\,(2m_0 - y\not{p}) \cdot \ln\frac{y\Lambda^2}{(1-y)m_0^2 - y(1-y)p^2 + y\lambda^2}$$

log has a branch cut for $\underline{p^2 \geq (m_0+\lambda)^2}$

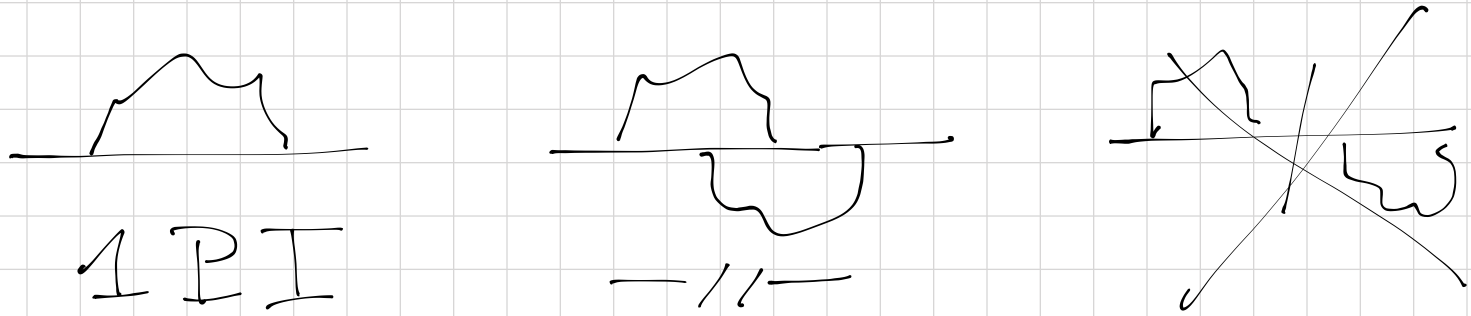
Denom. vanishes for

$$y_\pm = \frac{p^2+m_0^2-\lambda^2}{2p^2} \pm \frac{\sqrt{[p^2-(m_0+\lambda)^2][p^2-(m_0-\lambda)^2]}}{2p^2}$$

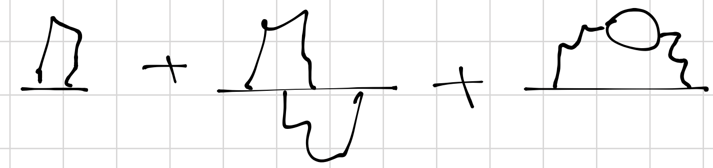
$\uparrow$ 2-part. 3-mom. in c.m.

To find the simple pole $\rightarrow$ abandon perturbative exp.

Define 1-particle irreducible contr. (no single-particle cuts)



1PI —//—

$$-i\Sigma(p) = \sum -\!\!\left(1PI\right)\!\!- = $$  $+ \ldots$

(not Σ_2)

$$\int d^4x\, e^{ipx} \langle \Omega | T\psi(x)\bar{\psi}(0)|\Omega\rangle$$

$$= \text{———} + -(1PI)- + -(1PI)-(1PI)- + \ldots$$

$$\frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\frac{\Sigma(\not{p})}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^2$$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

$$= \frac{i}{\not{p}-m_0-\Sigma(\not{p})}$$

Physical mass $m \to$ solution of

$$\not{p}-m_0-\Sigma(\not{p}) = 0 \Big|_{\not{p}=m}$$

$$Z_2^{-1} = 1 - \frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m}$$

Order α

$$\delta m = m - m_0 = \Sigma_2(\not{p}=m) \overset{\alpha \ll 1}{\simeq} \Sigma_2(\not{p}=m_0)$$

$$\delta Z_2 = Z_2 - 1 = \frac{dZ_2}{d\not{p}}\Big|_{\not{p}=m}$$

$$m_0 \simeq m$$

$$= \frac{\alpha}{2\pi}\int_0^1 dy \left[-y \ln \frac{y\lambda^2}{(1-y)^2 m^2 + y\lambda^2} + \frac{2m^2\, y(1-y)(2-y)}{(1-y)^2 m^2 + y\lambda^2} \right]$$

$\underline{\delta Z_2 + \delta F_1(0)}$

$$\delta F_1(0) = \frac{\alpha}{2\pi}\int_0^1 dy\,(1-y) \left[\ln \frac{y\lambda^2}{(1-y)^2 m^2 + y\lambda^2} + \frac{(1-4y+y^2)\,m^2}{(1-y)^2 m^2 + y\lambda^2} \right]$$

$\ln\lambda^2$: $\frac{\alpha}{2\pi}\int_0^1 dy(1-2y) = \frac{\alpha}{2\pi}\, y(1-y)\Big|_0^1 = 0$

$\ln y$: $\frac{\alpha}{2\pi}\int_0^1 dy(1-2y)\ln y = -\frac{\alpha}{2\pi}\int_0^1 dy(1-y) = -\frac{\alpha}{4\pi}$

$$\frac{m^2(1-y)\left[\cancel{2}y(\cancel{2}-y) + 1 - \cancel{4}y + \cancel{y^2}\right]}{(1-y)^2 m^2 + \cancel{y\lambda^2}} \quad = (1-y)(1+y)$$

$$= (1+y)$$

const. : $\frac{\alpha}{2\pi}\int_0^1 dy(1+y) = \frac{3\alpha}{4\pi}$

$$-\int_0^1 dy\,(1-2y)\ln\left((1-y)^2 m^2 + y\lambda^2\right)$$

$$= \cancel{-y(1-y)\ln\Big|_0^1} + \int_0^1 dy\, y\cancel{(1-y)} \frac{-2\cancel{(1-y)m^2} + \cancel{\lambda^2}}{\cancel{(1-y)^2 m^2} + \cancel{y\lambda^2}}$$

0

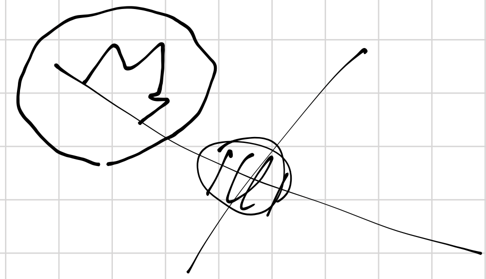
$$= \int_0^1 dy(-2y) = -1 \longrightarrow -\frac{\alpha}{2\pi}$$

Putting all together:

$$\boxed{\delta Z_2 + \delta F_1(0) = 0}$$

LSZ reduction

We have applied Feynman rules to <u>amputated</u> diagrams



With mass renorm.
this arbitrary choice
finally makes sense:
we operate w. physical mass that includes all such terms

Interacting th. $m_0 \longrightarrow m$
But there are also cuts in $\rho(M^2)$

$$\langle f | S | i \rangle = \left[\int d^4x_1\, e^{-ip_1x_1} \underline{(\Box_1^2 + m^2)}\right] \cdots \left[\int d^4x_n\, e^{ip_nx_n} (\Box + m^2)\right]$$

$$\longrightarrow \langle \Omega | T\, \phi(x_1) \dots \phi(x_n) | \Omega \rangle$$

$$\Box + m^2 \longrightarrow -p_i^2 + m^2$$

Asymptotic states $\quad p_i^2 = m^2$

$\Box + m^2$ sets to ϕ everything that has not a simple pole $\sim \frac{1}{p^2 - M^2} \longrightarrow$ justifies our naive Feynman rules