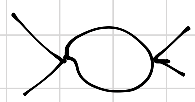
Lecture 24

Upon allowing for loop corrections $\rightarrow$
all ingredients (couplings (effective), masses, amplitudes) change

Loops lead to ∞

Renormalized PT $\rightarrow$ define $\lambda_R \equiv -\mathcal{M}(s_0)$ (ϕ^4 theory)

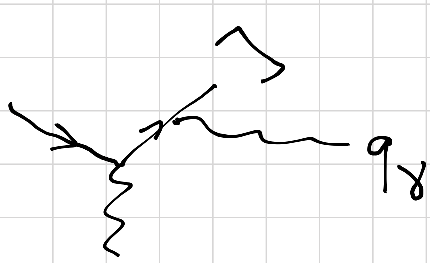


Theory predicts $\mathcal{M}(s) - \mathcal{M}(s_0)$

Today we discuss IR divergences

$$\mathcal{M}_0 = -\frac{e^2}{t}\,\bar{u}(k')\gamma^\mu u(k)\cdot J_\mu$$

Experiment: prepare an e^- beam
put a detector to measure e^-



The real γ carries away ω_γ
Detector has a finite E-res.
For $\omega_\gamma < \Delta E$ we cannot distinguish $ep \to ep$ from $ep \to ep\gamma$

$\Rightarrow$ # of events $\sim \dfrac{d\sigma_0}{d\Omega} + \displaystyle\int d\omega_\gamma \frac{d\sigma_{1\gamma}}{d\Omega\, d\omega_\gamma}$

The potential problem: γ is massless
There is no mass gap to distinguish e^-, $e\gamma$

$$\mathcal{M}^{1\gamma} = -\frac{e^3}{t_{1\gamma}}\left[\frac{\bar{u}(k')\,\not{\varepsilon}^*(q_\gamma)\,(\not{k}'+\not{q}_\gamma+m)\,\gamma^\mu u(k)}{(k'+q_\gamma)^2-m^2} + \frac{\bar{u}(k')\,\gamma^\mu(\not{k}-\not{q}_\gamma+m)\,\not{\varepsilon}^*(q_\gamma)\,u(k)}{(k-q_\gamma)^2-m^2}\right] J_\varphi$$

<u>Low ω_γ behavior</u>

$$q_\gamma \to 0$$

$$\bar{u}(k')\,\not{\varepsilon}^*(\not{k}'+\cancel{\not{q}_\gamma}+m) \cong 2(k'\varepsilon^*)\,\bar{u}(k') + O(q_\gamma)$$

$\downarrow$

$$\mathcal{M}^{1\gamma} = \mathcal{M}^0 \cdot e\left[\frac{(k'\varepsilon^*)}{(k'q_\gamma)} - \frac{(k\varepsilon^*)}{(kq_\gamma)}\right]$$

$$\frac{d\sigma^{1\gamma}}{d\Omega} = \frac{d\sigma_0}{d\Omega}\cdot e^2 \int \frac{d^3q_\gamma}{(2\pi)^3}\,\frac{1}{2\omega_\gamma}\sum_{\lambda_\gamma}\left|\frac{k'\varepsilon^*}{k'q_\gamma} - \frac{k\varepsilon^*}{kq_\gamma}\right|^2$$

$$\sum_{\lambda_\gamma}\varepsilon^\nu_{\lambda_\gamma}\varepsilon^{*\nu'}_{\lambda_\gamma} = \underline{-}\,g^{\nu\nu'} \qquad \alpha = \frac{e^2}{4\pi}$$

$$\hookrightarrow \quad \delta^{1\gamma} = \frac{d\sigma^{1\gamma}}{d\sigma_0} = \frac{\alpha}{\pi}\int d\omega_\gamma\,\omega_\gamma \int \frac{d\Omega_\gamma}{4\pi}$$

$$\times(-1)\left|\frac{k'^\mu}{(k'q_\gamma)} - \frac{k^\mu}{(kq_\gamma)}\right|^2 \leftarrow \frac{1}{\omega_\gamma^2}$$

$$\delta^{1\gamma} = \frac{\alpha}{\pi}\int d\omega_\gamma\,\omega_\gamma \int \frac{d\Omega_\gamma}{4\pi}\left[\frac{2kk'}{(k'q_\gamma)(kq_\gamma)} - \frac{m^2}{(k'q_\gamma)^2} - \frac{m^2}{(kq_\gamma)^2}\right]$$

Pick a frame $E = E'$ <u>c.m.</u> or Breit

$$k^\mu = (E, 0, 0, \underline{k}) \qquad k'^\mu = (E, \underline{k\sin\theta}, 0, \underline{k\cos\theta})$$

$$q_\gamma^\mu = \omega_\gamma (1, \hat{q}_\gamma) \rightarrow (\theta_\gamma, \varphi)$$

$$\int \frac{d\Omega_\gamma}{4\pi} \frac{m^2}{(k q_\gamma)^2} = \frac{m^2}{\omega_\gamma^2} \underbrace{\int_0^{2\pi} \frac{d\varphi}{2\pi}}_{= 1} \underbrace{\frac{1}{2}\int_{-1}^{1} \frac{d\cos\theta_\gamma}{(E - k\cos\theta_\gamma)^2}}_{= \frac{1}{2k} \frac{1}{E - k\cos\theta}\Big|_{-1}^{1} = \frac{1}{m^2}}$$

$$= \frac{1}{\omega_\gamma^2}$$

Ex.

$$\frac{1}{4\pi} \int \frac{d\Omega_\gamma}{(k q_\gamma)(k' q_\gamma)} = \frac{1}{4\pi} \int \frac{d\Omega_\gamma}{(E - \vec{k}\hat{q}_\gamma)(E - \vec{k}'\hat{q}_\gamma)}$$

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2}$$

$$E - \vec{k}(1-x)\hat{q}_\gamma - x\vec{k}'\hat{q}_\gamma = E - \vec{k}_x \hat{q}_\gamma$$

Define the $\hat{e}_z$: $\parallel$ to $\vec{k}_x$

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{2} \int_{-1}^{1} \frac{d\cos\theta_\gamma}{(E - |\vec{k}_x| \cos\theta_\gamma)^2} = \frac{1}{E^2 - \vec{k}_x^2}$$

$$\vec{k}_x^2 = |\vec{k}(1-x) + x\vec{k}'|^2 \overset{!}{=} k^2 + x(1-x)t$$

$$t = (k - k')^2 = -(\vec{k} - \vec{k}')^2$$

$$2kk' = 2m^2 - t$$

$$\Downarrow$$

$$\delta^{1\gamma} = \frac{\alpha}{\pi} \int_0^{\Delta E} \frac{d\omega_\gamma}{\omega_\gamma} \int_0^1 dx \left[\frac{2m^2 - t}{m^2 - x(1-x)t} - 2 \right]$$

$$t \gg m^2 \qquad m \to 0 \qquad \int_0^1 \frac{dx}{x(1-x)} \longrightarrow \infty$$

$$\hookrightarrow \; 2\int_0^1 dx \, \frac{-t}{m^2 - xt} = 2 \ln\left(-\frac{t}{m^2}\right)$$

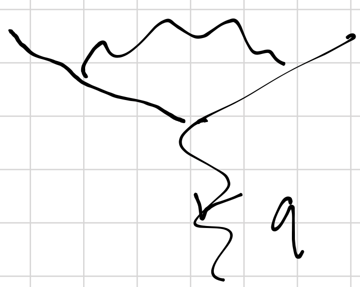
$$\int_0^{\Delta E} \frac{d\omega_\gamma}{\omega_\gamma} = \ln \omega_\gamma \Big|_0^{\Delta E} \to \infty$$

Let's put a finite mass gap $\to m_\gamma = \lambda$

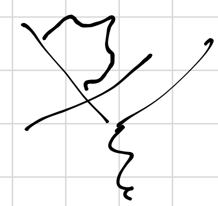
$$\int_\lambda^{\Delta E} \frac{d\omega_\gamma}{\omega_\gamma} = \frac{1}{2} \ln\left(\frac{\Delta E^2}{\lambda^2}\right)$$

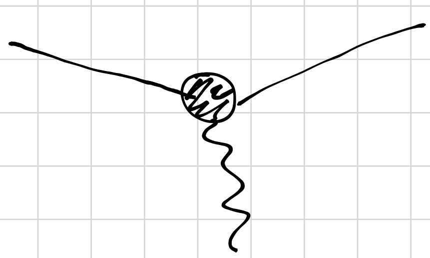
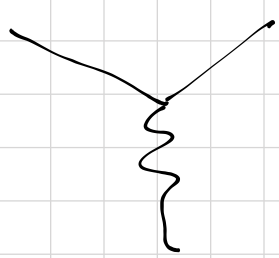
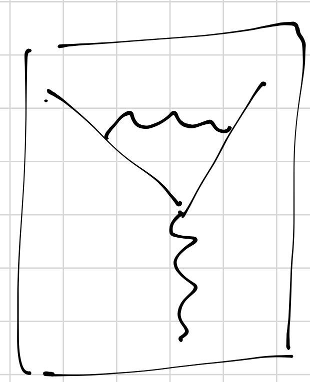
$$\boxed{\delta^{1\gamma} = \frac{\alpha}{\pi} \ln\left(\frac{\Delta E^2}{\lambda^2}\right) \ln\left(-\frac{t}{m^2}\right)}$$

We need to combine with something else containing $\log(\lambda) \to$ hope to cancel.



$\delta e \leftarrow$ 



 $=$  $+$  $+ \dots$

$-ie\,\bar{u}\,\Gamma^\mu u$ $\qquad$ $\underline{-ie\,\bar{u}\,\gamma^\mu u}$ $\qquad$ ....

$$\Gamma^\mu = F_1(q^2)\gamma^\mu - F_2(q^2)\,\frac{i\sigma^{\mu\alpha} q_\alpha}{2m}$$

tree level : $F_1(q^2) = 1$, $F_2(q^2) = 0$

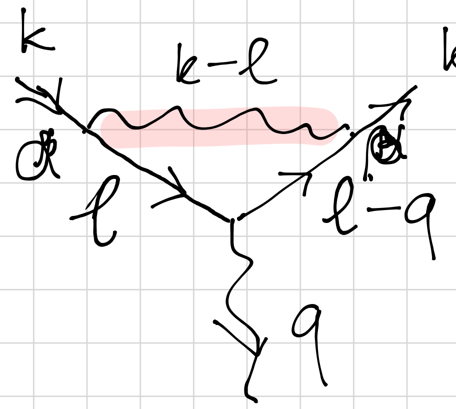
$O(\alpha)$: $1 + \delta F_1(q^2)$, $\delta F_2(q^2)$

Renorm. PT : $\Gamma^\mu(q^2=0) = \gamma^\mu$

$\underline{e_R}$ $\quad$ $F_1(q^2=0) = 1$

Definition $\quad \hookrightarrow \underline{\delta F_1(q^2=0) = 0}$

$$-ie\,\Gamma^\mu_{1-loop} = (-ie)^3 \int \frac{d^4 l}{(2\pi)^4} \frac{-i g_{\alpha\beta}}{(k-l)^2-\lambda^2+i\varepsilon} \frac{i}{l^2-m^2+i\varepsilon} \frac{i}{(l-q)^2-m^2+i\varepsilon}$$



$$\bar u(k')\gamma^\beta(\not l-\not q+m)\gamma^\mu(\not l+m)\gamma^\alpha u(k)$$

$$= -e^3 \int \frac{d^4 l}{(2\pi)^4} \frac{\bar u\,\gamma^\alpha(\not l-\not q+m)\gamma^\mu(\not l+m)\gamma_\alpha u}{\underbrace{[l^2-m^2+i\varepsilon]}_{A}\underbrace{[(l-q)^2-m^2+i\varepsilon]}_{B}\underbrace{[(l-k)^2-\lambda^2+i\varepsilon]}_{C}}$$

1. Feynman parameters

$$\frac{1}{ABC} = \Gamma(3)\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)\frac{1}{[xA+yB+zC]^3}$$

$\Gamma(n) = (n-1)!$

$A = l^2 - m^2 + i\varepsilon$

$B = l^2 - m^2 + i\varepsilon \underline{-2lq} + q^2$

$C = l^2 + m^2 + i\varepsilon \underline{-2lk} - \lambda^2$

$\hookrightarrow xA + yB + zC = \underline{l^2} + i\varepsilon - m^2(x+y-z) - z\lambda + yq^2$

$\qquad - \underline{2y\,lq} - \underline{2z\,lk}$

$\tilde l^2 - \Delta + i\varepsilon \qquad = (l - yq - zk)^2 - \Delta + i\varepsilon$

$$\Delta = (1-z)^2 m^2 + z\lambda^2 - yxq^2$$

$\delta(1-x-y-z)$

$$\int \frac{d^4 l}{(2\pi)^4} \frac{N}{[(l-yq-zk)^2-\Delta^2+i\varepsilon]^3} \qquad l \to \tilde l = l - yq - zk$$

$$N = \bar{u}(k')\,\gamma^\alpha(\not{\ell}-\not{q}+m)\gamma^\mu(\not{\ell}+m)\gamma_\alpha\, u(k)$$

$$\ell \Rightarrow \tilde{\ell}+yq+zk$$

$$\longrightarrow \int \frac{d^4\ell}{(2\pi)^4}\,\frac{\ell^\mu}{[\ell^2-\Delta+i\varepsilon]^3} = 0$$

$$\int \frac{d^4\ell}{(2\pi)^4}\,\frac{\ell^\mu \ell^\nu}{[\ell^2-\Delta+i\varepsilon]^3} = \frac{1}{4}g^{\mu\nu}\int \frac{\ell^2}{[\ell^2-\Delta+i\varepsilon]^3}$$

$$\Rightarrow N = -2\,\bar{u}(k')\Big[(\tilde{\not{\ell}}+z\not{k}+y\not{q})\gamma^\mu(\tilde{\not{\ell}}+z\not{k}-(1-y)\not{q})$$
$$+\, m^2\gamma^\mu$$
$$-2m\,(\underbrace{2\tilde{\ell}}_{\to 0}+2zk-(1-2y)q)^\mu\Big]\,u(k)$$

Ex.

$$= -2\,\bar{u}(k')\Big[\tilde{\not{\ell}}\gamma^\mu\tilde{\not{\ell}} + (z\not{k}+y\not{q})\gamma^\mu(z\not{k}-(1-y)\not{q})$$
$$+\, m^2\gamma^\mu - 2m\,(2zk-(1-2y)q)^\mu\Big]u(k)$$

$$\tilde{\not{\ell}}\gamma^\mu\tilde{\not{\ell}} = \underbrace{\tilde{\ell}_\alpha\tilde{\ell}_\beta}_{\to \frac{1}{4}g_{\alpha\beta}\tilde{\ell}^2}\cdot\gamma^\alpha\gamma^\mu\gamma^\beta = -\frac{1}{2}\tilde{\ell}^2\gamma^\mu$$

Next steps:

Use $\not{k}\,u(k) = m\,u(k)$

$\bar{u}(k')\,\not{q}\,u(k) = \bar{u}(k')(\not{k}-\not{k}')u(k) = 0$

$\Downarrow$

$$[\,...\,] = -\frac{1}{2}\tilde{\ell}^2\gamma^\mu + z^2 m\Big[\overbrace{(k+k')^\mu}^{=2k^\mu}+q^\mu\Big] + (1-z^2)m^2\gamma^\mu$$
$$+\, y(1-y)q^2\gamma^\mu - 2mz\big[(k+k')^\mu+q^\mu\big]$$
$$+\, 2m(1-2y)q^\mu + yzm\,(\underbrace{q^\mu+i\sigma^{\mu\alpha}q_\alpha}_{=\not{q}\gamma^\mu})$$

$$+(1-y)z\left[q^2\gamma^\mu - m\underbrace{\left(q^\mu - i\sigma^{\mu\alpha}q_\alpha\right)}_{"\gamma^\mu \not{q}"}\right]$$

$$\bar{u}(k')\gamma^\mu u(k) = \bar{u}(k')\left(\frac{(k+k')^\mu}{2m} - \frac{i\sigma^{\mu\alpha}q_\alpha}{2m}\right)u(k)$$

$\underline{q = k - k'}$ Gordon ID

$\Downarrow$

$$\Gamma = -2[\,\ldots\,] = -2\gamma^\mu\left[-\frac{1}{2}\tilde{\ell}^2 + m^2(1-4z+z^2) + (1-x)(1-y)q^2\right]$$

$$-2mq^\mu(2-z)(x-y) \rightarrow 0$$

$$+2mz(1-z)\,i\sigma^{\mu\alpha}q_\alpha$$

q^μ term: anti-sym. $x \Leftrightarrow y$

$$\left[\int_0^1 dx\,dy\,dz\;\delta(1-x-y-z)\qquad \Delta.\right.$$

symmetric $x \Leftrightarrow y$

$$\Gamma^\mu = 2e^3\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta+i\varepsilon]^3}\int_0^1 dx\,dy\,dz\;\delta(1-x-y-z)$$

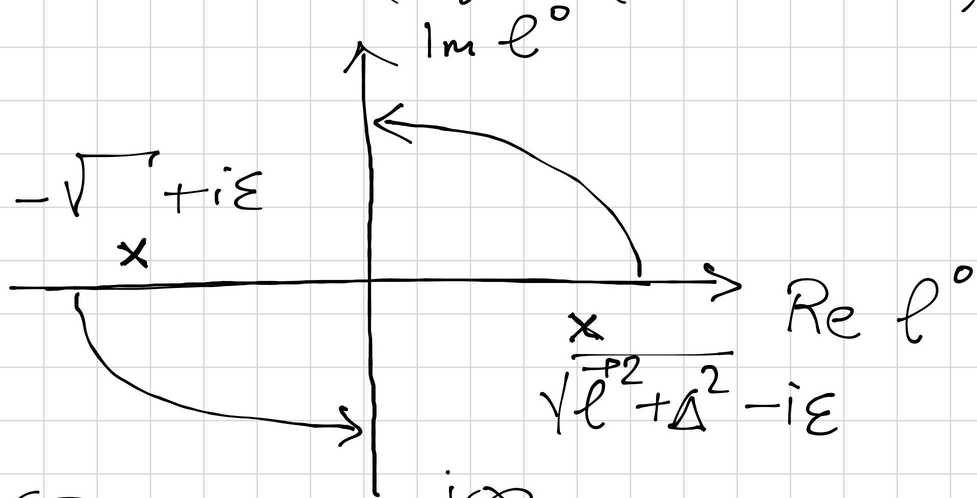
$$\cdot\left\{\gamma^\mu\left[-\frac{1}{2}\ell^2 + m^2(1-4z+z^2) + (1-x)(1-y)q^2\right]\right.$$

$$\left.-mz(1-z)\,i\sigma^{\mu\alpha}q_\alpha\right\}$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell^2)^m}{[\ell^2-\Delta+i\varepsilon]^n} \equiv I(m,n)$$

$$\Delta > 0$$

$$\ell^2-\Delta+i\varepsilon = (\ell^0)^2-(\vec{\ell}^2+\Delta-i\varepsilon)$$



$$\ell^0 \longrightarrow i\ell_E^0$$

Wick's rotation

$$\int_{-\infty}^{\infty} d\ell^0 = i\int_{-i\infty}^{i\infty} d\ell_E^0$$

$$-\ell_E^2 = (i\ell_E)^2-\vec{\ell}^2 = -(\ell_E^{02}+\vec{\ell}^2) < 0$$

$$I(m,n) = (-1)^{n+m}\, i\int \frac{d^4\ell_E}{(2\pi)^4} \frac{(\ell_E^2)^m}{[\ell_E^2+\Delta-i\varepsilon]^n}$$

Spherical coordinates in 4D

$$\int d^4\ell_E = \int_0^\infty \ell_E^3\, d\ell_E \underbrace{\int_0^\pi \sin^2\psi\, d\psi \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi}_{\int d\Omega_4}$$

$$\int d\Omega_4 = 2\pi^2$$

$$\ell_E^3\, d\ell_E = \frac{1}{2}\ell_E^2\, d\ell_E^2$$

$$x = \ell_E^2/\Delta$$

$$I(m,n) = \frac{i}{(4\pi)^2}(-1)^{m+n}\int_0^\infty \frac{dx\, x^{m+1}(1+x)^{-n}}{\Delta^{n-m-2}}$$

$$\int_0^\infty dx \frac{x^{m+1}}{(1+x)^n} = \frac{\Gamma(n-m-2)\,\Gamma(m+2)}{\Gamma(n)}$$

$$\boxed{I(m,n) = \frac{i}{(4\pi)^2}(-1)^{m+n}\frac{\Gamma(n-m-2)\,\Gamma(m+2)}{\Gamma(n)\;\Delta^{n-m-2}}}$$

We have 2 kinds of $\int$

$I(1,3)$ $\qquad \underline{\Gamma(3-1-2) = \Gamma(0) = \infty}$

$I(0,3)$ is finite

$$\Delta = (1-z)^2 m^2 + z\underline{\lambda^2} - xy\,q^2$$

$$I(1,3) \longrightarrow \frac{d}{d\lambda^2} I(1,3)$$

$$\frac{d}{d\lambda^2}\Delta^{-(n-m-2)} = -z\cdot\Delta^{-(n-m-1)}(n-m-2)$$

$$\frac{d}{d\lambda^2}\frac{\Gamma(n-m-2)}{\Delta^{n-m-2}} = -z\frac{(n-m-2)\,\Gamma(n-m-2)}{\Delta^{n-m-1}}$$

$$= -z\frac{\Gamma(1)}{\Delta} = \frac{-z}{\Delta}$$

$$I(1,3) = \int d\lambda^2\left(\frac{-z}{\Delta}\frac{i}{(4\pi)^2}\right)$$

$$= \frac{-iz}{(4\pi)^2}\int_{\Lambda^2}^{\lambda^2}\frac{d\lambda^2}{(1-z)^2m^2 - xyq^2 + z\lambda^2}$$

$$= \frac{i}{(4\pi)^2}\ln\left(\frac{\Delta_\Lambda}{\Delta}\right)$$

$$\Delta_\Lambda = (1-z)^2 m^2 - xy\,q^2 + z\Lambda^2$$

will follow it through later

$\Lambda^2 \gg$

UV-cutoff