Exercise sheet 11
Theoretical Physics 3 : QM WS2020/2021
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Exercise 1. Hyperfine splitting (50 points)

The electron and proton are both magnetic dipoles with magnetic moments

$$\begin{aligned} \vec{\mu}_e &= -g_e \frac{e}{2m_e}\vec{\mathbf{s}}_e\,, \quad g_e = 2\,, \\ \vec{\mu}_p &= +g_p \frac{e}{2m_p}\vec{\mathbf{s}}_p\,, \quad g_p = 5.59\,. \end{aligned}$$

As in classical electromagnetism, the magnetic field due to a point magnetic dipole of the proton is

$$\vec{\mathbf{B}}_p = \frac{\mu_0}{4\pi r^3}\left[3(\vec{\mu}_p\cdot\vec{\mathbf{e}}_r)\vec{\mathbf{e}}_r - \vec{\mu}_p\right] + \frac{2\mu_0}{3}\delta^{(3)}(\vec{\mathbf{r}})\vec{\mu}_p\,,$$

where $\vec{\mathbf{e}}_r = \vec{\mathbf{r}}/r$.

The electron magnetic dipole moment $\vec{\mu}_e$ interacts with the magnetic field generated by the proton. The corresponding contribution to the Hamiltonian is

$$\hat{H} = -\vec{\mu}_e\cdot\vec{\mathbf{B}}_p\,.$$

Consider $l = 0$ state of Hydrogen, so that the coupling between the proton and the magnetic field generated by the electron's orbital motion is absent. In this case the Hamiltonian

$$\hat{H} = \frac{\mu_0}{8\pi}\frac{g_p e^2}{m_p m_e}\frac{1}{r^3}\left[3(\vec{\mathbf{s}}_p\cdot\vec{\mathbf{e}}_r)(\vec{\mathbf{s}}_e\cdot\vec{\mathbf{e}}_r) - \vec{\mathbf{s}}_p\cdot\vec{\mathbf{s}}_e\right] + \frac{\mu_0 g_p e^2}{3m_p m_e}\vec{\mathbf{s}}_p\cdot\vec{\mathbf{s}}_e\delta^{(3)}(\vec{\mathbf{r}})\,,$$

fully describes the magnetic dipole interaction of the proton and electron.

According to the first order perturbation theory, the corresponding energy shift is

$$E = \langle\hat{H}\rangle = E_r + E_c\,,$$

where E_r and E_c are regular and contact contributions corresponding to the first and second terms in Hamiltonian respectively.

a) *(5 p.)* Explain why E_r vanishes for $l = 0$ state.

b) *(15 p.)* Show that the energy shift E_c in first order perturbation theory, is given by

$$E_c = \frac{\mu_0 e^2 g_p}{3m_e m_p}\frac{1}{2}(s^2 - s_e^2 - s_p^2)|\psi_{n00}(0)|^2,$$

where $\vec{\mathbf{s}} = \vec{\mathbf{s}}_e + \vec{\mathbf{s}}_p$ and ψ is the hydrogen spatial wave function.

c) *(5 p.)* Consider the ground state. How many levels are there? What is the degeneracy of each level?

d) *(15 p.)* Derive the energy shifts for both singlet and triplet ground states. Find the numerical value of the hyperfine splitting between them and the corresponding wavelength.

e) *(10 p.)* Compare the magnitudes of the hyperfine splitting of the ground state and the fine splitting of the $n = 2$ states. What is the main reason that the hyperfine splitting is smaller than the fine splitting? Give an order of magnitude estimate of their ratio.

Exercise 2. Magnetic resonance (50 points)

The neutron magnetic moment $\vec{\mu}_n$ interacts with an external magnetic field $\vec{\mathbf{B}}$ through the (interaction) Hamiltonian:

$$\hat{H} = -\vec{\mu}_n \cdot \vec{\mathbf{B}}\,.$$

The neutron magnetic moment vector is expressed in terms of the neutron spin-1/2 vector as: $\vec{\mu}_n = \gamma_n \frac{\hbar}{2}\vec{\sigma}$, where $\gamma_n < 0$ is the neutron gyromagnetic ratio, and $\vec{\sigma}$ are the Pauli matrices.

a) *(10 p.)* Consider first the case of a constant magnetic field

$$\vec{\mathbf{B}} = B_0 \hat{\mathbf{e}}_z\,,$$

with $\hat{\mathbf{e}}_z$ the unit vector along the z-axis, and B_0 a constant.

Write down the Hamiltonian $\hat{H}$ in 2×2 matrix form for this case, using the notation: $\omega_0 \equiv -\gamma_n B_0$ (*Larmor frequency*) and determine the eigenvalues.

b) *(5 p.)* At time t_0 the neutron enters a cavity and feels, in addition to the constant magnetic field along the z-direction, a rotating component in the x, y plane as:

$$\vec{\mathbf{B}} = B_1 \cos\omega t\,\hat{\mathbf{e}}_x + B_1 \sin\omega t\,\hat{\mathbf{e}}_y + B_0\hat{\mathbf{e}}_z\,,$$

where ω is the (externally controlled) frequency of the rotating field, and B_1 is its magnitude.

Write down the Hamiltonian $\hat{H}$ in 2×2 matrix form for that case, using the notation $\omega_0 \equiv -\gamma_n B_0$ and $\omega_1 \equiv -\gamma_n B_1$.

c) *(10 p.)* The neutron spin 1/2 state at a given time t is given (in matrix notation) by:

$$\Psi(t) = \begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix}\,,$$

where $c_+(t)$ and $c_-(t)$ are the amplitudes to be in the spin-up and spin-down states respectively.

Determine the time evolution of $c_\pm(t)$ for $t \geq t_0$ using the time dependent Schroedinger equation: $\hat{H}\Psi(t) = i\hbar\frac{\partial}{\partial t}\Psi(t)$.

d) *(15 p.)* Write down the differential equations for $c_+(t)$ and $c_-(t)$ and solve these equations for the resonance condition $\omega = \omega_0$.

Hint: Express

$$\begin{aligned} c_+(t) &= e^{-\frac{i}{2}\omega_0(t-t_0)}\beta_+(t)\,, \\ c_-(t) &= e^{+\frac{i}{2}\omega_0(t-t_0)}\beta_-(t)\,, \end{aligned}$$

and write down the equivalent differential equations for $\beta_\pm(t)$, which can be easily solved. Express the solution as function of the time difference $(t - t_0)$, and as function of the initial conditions $c_+(t_0)$ and $c_-(t_0)$.

e) *(10 p.)* The obtained solution allows to express the neutron spin state just after traversing the cavity, *i.e.* at a time t_1, as:

$$\begin{pmatrix} c_+(t_1) \\ c_-(t_1) \end{pmatrix} = U(t_1, t_0) \begin{pmatrix} c_+(t_0) \\ c_-(t_0) \end{pmatrix}\,,$$

where the 2×2 matrix U can be written as:

$$U(t_1, t_0) = \begin{pmatrix} e^{-i\chi}\cos\phi & -ie^{-i\delta}\sin\phi \\ -ie^{+i\delta}\sin\phi & e^{+i\chi}\cos\phi \end{pmatrix}\,,$$

Use your above obtained solution to express χ, δ, and ϕ in terms of only $\omega_0, \omega_1, (t_1 - t_0)$, and $(t_1 + t_0)$.