Relativistic QFT (Theo 6a): Exercise Sheet 12
Total: 100 points

06/02/2021

1. Vertex correction in QED (50 points)

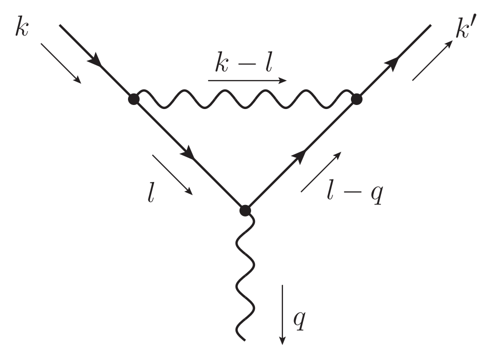

Figure 1: Feynman diagram for the one-loop vertex correction in QED.

Consider the matrix expression

$$\mathcal{N}(l,q) = \bar{u}(k')\gamma^\alpha(\not{l} - \not{q} + m)\gamma^\mu(\not{l} + m)\gamma_\alpha u(k) \tag{1}$$

in the numerator of the matrix element of the one-loop vertex correction in QED,

$$\bar{u}(k')\Gamma^\mu(q)u(k) = \frac{e^2}{i}\int \frac{d^4l}{(2\pi)^4}\frac{\mathcal{N}(l,q)}{[l^2 - m^2 + i\epsilon][(l-q)^2 - m^2 + i\epsilon][(l-k)^2 - \lambda^2 + i\epsilon]}. \tag{2}$$

In order to take the integral, it is convenient to rewrite the denominator using the Feynman parametrization,

$$\begin{aligned}\bar{u}(k')\Gamma^\mu(q)u(k) = \frac{2e^2}{i}\int_0^1 dx\int_0^1 dy\int_0^1 dz\delta(1-x-y-z)\\ \int \frac{d^4l}{(2\pi)^4}\frac{\mathcal{N}(l,q)}{\Big\{x[l^2 - m^2 + i\epsilon] + y[(l-q)^2 - m^2 + i\epsilon] + z[(l-k)^2 - \lambda^2 + i\epsilon]\Big\}^3} =\\ = \frac{2e^2}{i}\int_0^1 dx\int_0^1 dy\int_0^1 dz\delta(1-x-y-z)\int \frac{d^4l}{(2\pi)^4}\frac{\mathcal{N}(l,q)}{\Big\{(l - yq - zk)^2 - \Delta + i\epsilon\Big\}^3},\end{aligned} \tag{3}$$

where $\Delta = (1-z)^2m^2 + z\lambda^2 - yxq^2$.

(a) In order to make the integration over the loop momentum l convenient, one should change the variable $l \to \tilde{l} = l - yq - zk$, so $l = \tilde{l} + yq + zk$.

- Write the expression for the numerator $\mathcal{N}(\tilde{l}, q)$ and simplify it using the identities

$$\gamma^\alpha \gamma_\alpha = 4, \quad \gamma^\alpha \gamma^\mu \gamma_\alpha = -2\gamma^\mu, \quad \gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4\gamma^\mu \gamma^\nu, \quad \gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\alpha = -2\gamma^\rho \gamma^\nu \gamma^\mu. \tag{4}$$

- Get the midway result for the numerator, given in lecture,

$$\mathcal{N}(\tilde{l}, q) = -2\bar{u}(k')\Big[\tilde{\not{l}}\gamma^\mu \tilde{\not{l}} + (z\not{k} + y\not{q})\gamma^\mu(z\not{k} - (1-y)\not{q}) + m^2\gamma^\mu - 2m(2zk^\mu - (1-2y)q^\mu)\Big]u(k) \tag{5}$$

omitting the terms with even power of momentum $\tilde{l}$ since it gives zero contribution to the integral.

(b) Using the fact that the initial and final electron lies on mass shell, simplify (5), drugging all possible $\not{k}$ to the right, and all possible $\not{k}'$ to the left and applying the Dirac equation,

$$\not{k}u(k) = mu(k), \quad \bar{u}(k')\not{k}' = \bar{u}(k')m, \quad \bar{u}(k')(\not{k} - \not{k}')u(k) = \bar{u}(k')\not{q}u(k) = 0. \tag{6}$$

(c) Simplify the obtained result by using the following identities

$$\not{q}\gamma^\mu = q^\mu + i\sigma^{\mu\alpha}q^\alpha, \quad \gamma^\mu \not{q} = q^\mu - i\sigma^{\mu\alpha}q^\alpha, \tag{7}$$

$$2k^\mu = (k + k')^\mu + q^\mu, \quad 2k'^\mu = (k + k')^\mu - q^\mu, \tag{8}$$

and

$$\bar{u}(k')(k + k')^\mu u(k) = \bar{u}(k')\left[2m\gamma^\mu + i\sigma^{\mu\alpha}q_\alpha\right]u(k), \tag{9}$$

where the latter follows from the Gordon identity, and arrive at the final result for the numerator

$$\begin{aligned}\mathcal{N}(\tilde{l}, q) = \bar{u}(k')\Big\{ -2\gamma^\mu \Big[&-\frac{1}{2}\tilde{l}^2 + m^2(1 - 4z + z^2) + (1-x)(1-y)q^2 \\ &- 2mq^\mu(2-z)(x-y) \\ &+ 2mz(1-z)i\sigma^{\mu\alpha}q_\alpha\Big]\Big\}u(k)\end{aligned} \tag{10}$$

2. Soft photon emisson in electron-nucleon scatteting (50 points)

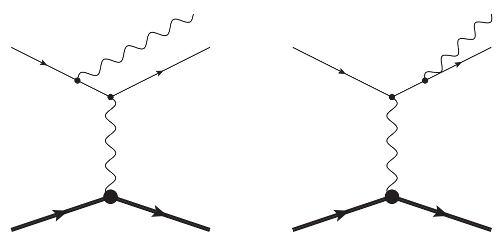

Figure 2: Feynman diagrams that contribute to the soft photon radiative corrections to the electron current in electron-nucleon scattering.

Consider the radiative corrections to the electron-nucleon scattering process, shown in Fig.2, due to the soft photon emission. Use the following kinematical conventions: let the initial momenta of the electron and the nucleon to be k and p respectively, the final momenta of the electron and the nucleon to be k' and p' respectively, and the final momentum of the soft photon to be q_γ with the energy ω_γ. The nucleon and the electron masses are M and m, respectively.

(a) Working to leading order in ω_γ express the soft-photon scattering amplitude $\mathcal{M}_{1\gamma}$ through the elastic amplitude (without a real photon in the final state) $\mathcal{M}_0$. Correspondingly, rewrite $\sum_{\text{spins}} |\mathcal{M}_{1\gamma}|^2$ via $\sum_{\text{spins}} |\mathcal{M}_0|^2$ in the approximation $\omega_\gamma \to 0$,

$$\sum_{\text{spins}} |\mathcal{M}_{1\gamma}|^2 = \sum_{\text{spins}} |\mathcal{M}_0|^2 \left(\frac{k'^\mu}{(k'q_\gamma)} - \frac{k^\mu}{(kq_\gamma)} \right)^2. \tag{11}$$

(b) As an intermediate step, assume the final photon solid angle to be spherical (will be elaborated at the next step) and perform the phase-space integration regularizing the $1/\omega_\gamma$ singularity by introducing a small but finite photon mass λ, and integrating up to a maximum photon energy $\Delta\tilde{E}$,

$$\frac{1}{4\pi}\int_\lambda^{\Delta\tilde{E}} \frac{d^3q_\gamma}{\omega_\gamma}\left(\frac{k'^\mu}{(k'q_\gamma)} - \frac{k^\mu}{(kq_\gamma)}\right)^2 \approx \ln\frac{\Delta\tilde{E}^2}{\lambda^2}\left[\ln\frac{-t}{m^2} - 1\right], \tag{12}$$

where the limit $t \gg m^2$ was taken.

(c) Express the differential cross section for soft photon emission $d\sigma_{1\gamma}$ via that for the tree-level process $d\sigma_0$ using the standard expression for the cross section for $2 \to n$ scattering,

$$d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(pk) - m^2M^2}} d\Phi_n(p+k; p_1, p_2, \dots p_n), \quad \text{with}$$

$$d\Phi_n(p+k; p_1, p_2, \dots p_n) = (2\pi)^4\delta^4(p + k - \sum_{i=1}^n p_i)\Pi_{i=1}^n \frac{d^3p_i}{2E_i}, \quad E_i = \sqrt{\vec{p}_i^2 + M_i^2} \tag{13}$$

Apply this formula with $d\Phi_2(p+k; p', k')$ to obtain the elastic differential cross section in the c.m. frame of $(p+k)^\mu = (\sqrt{s}, \vec{0})$,

$$\frac{d\sigma_0}{d\Omega_e} = \frac{1}{64\pi^2 s}\sum_{\text{spins}}|\mathcal{M}_0|^2. \tag{14}$$

Next apply the formula with $d\Phi_3(p+k; p', k', q_\gamma)$ to obtain the differential cross section with a soft real photon in the final state.
Hint: integrate over $d^3\vec{p}'$ with $\delta^4(p+k-p'-k'-q_\gamma) = \delta(E_p+E-E'_p-E'-\omega_\gamma)\delta^3(\vec{p}+\vec{k}-\vec{p}'-\vec{k}'-\vec{q}_\gamma)$ first; to evaluate the remaining energy δ-function work in the c.m. frame of $(p'+q^\gamma)^\mu = (p+k-k')^\mu = (w, \vec{0})$ Use a tilde to denote energies and momenta in that frame. In this frame the solid angle of $\vec{q}_\gamma$ is spherical and the result of the integration obtained in the previous step can be used. Also, $\delta(\tilde{E}_p + \tilde{E} - \tilde{E}'_p - \tilde{E}' - \tilde{\omega}_\gamma) = 2\tilde{E}'_p\delta(w^2 - M^2 - 2w\tilde{\omega}_\gamma)$.

Determine the value of the detector resolution $\Delta\tilde{E}$ via that in the c.m. frame ΔE. To that end, one has the identity $(p+k-k')^2 = (p'+q_\gamma)^2$. Evaluate the left hand side in the c.m. frame denoting $E' = E'^{\text{elastic}} - \Delta E$, with $E'^{\text{elastic}} = (s - M^2)/2\sqrt{s}$. Evaluate the right hand side in the c.m. frame of $p' + q_\gamma$, obtaining $M\Delta\tilde{E} = \sqrt{s}\Delta E$.

Now you can neglect $\tilde{\omega}_\gamma$ in the δ-function $\delta(w^2 - M^2 - 2w\tilde{\omega}_\gamma) \to \delta(w^2 - M^2)$ and go to the c.m. frame of $p + k$ using the delta function to integrate over the elastic electron energy E'. Putting all ingredients together obtain

$$\frac{d\sigma_{1\gamma}}{d\Omega_e} = \frac{d\sigma_0}{d\Omega_e}\frac{\alpha}{\pi}\ln\frac{s}{M^2}\frac{\Delta E^2}{\lambda^2}\left[\ln\frac{-t}{m^2} - 1\right]. \tag{15}$$