$\Longrightarrow$ VORLESUNG 19 QM

$$\bar{S}\ (S_x, S_y, S_z)$$

$$\begin{cases} S^2 = S_x^2 + S_y^2 + S_z^2 \\ S_z \end{cases}$$

$S = \frac{1}{2}$ SPIN 1/2

$\longrightarrow \quad |S\, S_z\rangle$

$$S^2 \left|\tfrac{1}{2}\ \pm\tfrac{1}{2}\right\rangle = \hbar^2 \frac{3}{4} \left|\tfrac{1}{2}\ \pm\tfrac{1}{2}\right\rangle$$

$$S_z \left|\tfrac{1}{2}\ \pm\tfrac{1}{2}\right\rangle = \pm\frac{\hbar}{2} \left|\tfrac{1}{2}\ \pm\tfrac{1}{2}\right\rangle$$

$$\longrightarrow \quad \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

$$\left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \quad \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \longleftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{S} = \frac{\hbar}{2}\,\bar{\sigma}$$

$\hookrightarrow$ PAULI MATRIZEN

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

ALLGEMEINEN SPIN ZUSTAND

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= a\,\underbrace{\chi_\uparrow}_{\text{SPIN UP}} + b\,\underbrace{\chi_\downarrow}_{\text{SPIN DOWN}}$$

$|a|^2$ WAHRSCHEINLICHKEIT $\uparrow$

$|b|^2$ " $\downarrow$

$$|a|^2 + |b|^2 = 1$$

$$\chi^\dagger \chi = 1$$

$$\begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1$$

↳ EIGENWERTE, EIGENZUSTÄNDE S_x?

- $$S_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2} \sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

⇝ EIGENWERTE

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0$$

$$\Downarrow$$

$$\lambda = \pm \frac{\hbar}{2}$$

⇝ EIGENZUSTÄNDE

$$S_x = +\frac{\hbar}{2}$$

$$S_x \, \chi_\uparrow^{(x)} = +\frac{\hbar}{2} \, \chi_\uparrow^{(x)}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\beta = +\alpha$$

$$\rightsquigarrow \quad \chi_\uparrow^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\chi^\dagger \chi = 1$$

$$S_x \, \chi_\uparrow^{(x)} = -\frac{\hbar}{2} \, \chi_\uparrow^{(x)}$$

$$\beta = -\alpha$$

$$\rightsquigarrow \quad \chi_\downarrow^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

BELIEBIGEN SPINOR

$$\chi = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= a \, \chi_{\uparrow} + b \, \chi_{\downarrow}$$

$$= \left(\frac{a+b}{\sqrt{2}} \right) \underbrace{\chi_{\uparrow}^{(x)}}_{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} + \left(\frac{a-b}{\sqrt{2}} \right) \underbrace{\chi_{\downarrow}^{(x)}}_{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$= \frac{1}{2} \begin{pmatrix} a+b + a-b \\ a+b - (a-b) \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\rightsquigarrow \underset{\uparrow}{|a|^2} , \quad \underset{\downarrow}{|b|^2} \qquad \text{z - AXE}$$

$$\rightsquigarrow \left| \frac{a+b}{\sqrt{2}} \right|^2 \quad \uparrow \qquad \text{x - AXE}$$

$$\left| \frac{a-b}{\sqrt{2}} \right| \quad \downarrow \qquad \text{x - AXE}$$

BEISPIEL

SPIN 1/2 $\qquad \chi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$

- $S_z \frown +\frac{h}{2}, \; -\frac{\hbar}{2}$?

$$\chi = a\,\chi_{\uparrow} + b\,\chi_{\downarrow}$$

$$a = \frac{1}{\sqrt{6}}(1+i) \longrightarrow P_{\uparrow} = \frac{1}{3}$$

$$b = \frac{2}{\sqrt{6}} \longrightarrow P_{\downarrow} = \frac{2}{3}$$

- $S_x \frown +\frac{h}{2}, \; -\frac{\hbar}{2}$?

$$\chi = \left(\frac{a+b}{\sqrt{2}}\right) \chi_{\uparrow}^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right) \chi_{\downarrow}^{(x)}$$

$$\frac{a+b}{\sqrt{2}} = \frac{1}{\sqrt{12}}(3+i) \longrightarrow P_{\uparrow}^{(x)} = \frac{5}{6}$$

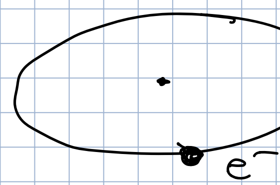
$$\frac{a-b}{\sqrt{2}} = \frac{1}{\sqrt{12}}(-1+i) \longrightarrow P_{\downarrow}^{(x)} = \frac{1}{6}$$

$\Rightarrow$ e^- in $\bar{B}$

$\bar{S}$

$\downarrow$

$\leadsto$ $\bar{\mu} = \gamma \, \bar{S}$



$\uparrow$

GYROMAGN
KONSTANTE

$$\left(\gamma = \frac{e}{2m} \, g \right)$$

SPIN $1/2$: DIRAC

$g = 2$

QFT $g = 2{,}0023\ldots$

$\uparrow$

$\leadsto$ in $\bar{B}$

$H = - \bar{\mu} \cdot \bar{B}$ $\qquad \bar{B}$

$\bar{\mu} \parallel \bar{B}$ $\qquad \uparrow \Uparrow$

$\bar{\mu}$ ANTI $\parallel$ $\bar{B}$ $\qquad \uparrow \Downarrow$

$$\rightsquigarrow \quad H = -\gamma\, \overline{S} \cdot \overline{B}$$

$$\overline{B} = B_0\, \overline{e}_z \qquad \text{KONSTANTES } B_0$$

$$H = -\gamma B_0\, S_z = -\gamma B_0 \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

EIGENZUSTÄNDE H

$$\chi_\uparrow \curvearrowright E_\uparrow = -\gamma B_0 \frac{\hbar}{2}$$

$$\chi_\downarrow \curvearrowright E_\downarrow = +\gamma B_0 \frac{\hbar}{2}$$

$$H\, \chi(t) = i\hbar \frac{\partial}{\partial t} \chi(t)$$

$\chi(t)$ SUMME DER STATIONÄREN ZUSTÄNDE

$$\chi(t) = a\,\chi_\uparrow\, e^{-\frac{i}{\hbar}E_\uparrow t} + b\,\chi_\downarrow\, e^{-\frac{i}{\hbar}E_\downarrow t}$$

$$\chi(t=0) = a\,\chi_\uparrow + b\,\chi_\downarrow = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$|a|^2 + |b|^2 = 1$$

$$\chi(t) = \begin{pmatrix} a \; e^{+\frac{i}{2}\gamma B_0 t} \\ b \; e^{-\frac{i}{2}\gamma B_0 t} \end{pmatrix}$$

$$a = \cos\frac{\alpha}{2}$$

$$b = \sin\frac{\alpha}{2}$$

$$S_x, \; S_y, \; S_z$$

$$\langle S_x \rangle = \chi^{+}(t) \; S_x \; \chi(t)$$

$$= \begin{pmatrix} \cos\frac{\alpha}{2} e^{-\frac{i}{2}\gamma B_0 t} & \sin\frac{\alpha}{2} e^{+\frac{i}{2}\gamma B_0 t} \end{pmatrix}$$

$$\cdot \underbrace{\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} \; e^{+\frac{i}{2}\gamma B_0 t} \\ \sin\frac{\alpha}{2} \; e^{-\frac{i}{2}\gamma B_0 t} \end{pmatrix}}$$

$$\begin{pmatrix} \sin\frac{\alpha}{2} \; e^{-\frac{i}{2}\gamma B_0 t} \\ \cos\frac{\alpha}{2} \; e^{+\frac{i}{2}\gamma B_0 t} \end{pmatrix}$$

$$= \frac{\hbar}{2} \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(e^{-i\gamma B_0 t} + e^{+i\gamma B_0 t}\right)$$

$$2\cos(\gamma B_0 t)$$

$$\langle S_x \rangle = \frac{\hbar}{2} \sin\alpha \, \cos(\underbrace{\gamma B_0}_{\omega = \gamma B_0} t)$$

LARMOR FREQUENZ

$$\langle S_y \rangle = \chi^{+}(t) \underbrace{S_y}_{\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}} \chi(t)$$

$$= \frac{\hbar}{2} \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} (-i) \underbrace{\left(e^{-i\gamma B_0 t} - e^{i\gamma B_0 t} \right)}_{-2i\sin(\gamma B_0 t)}$$

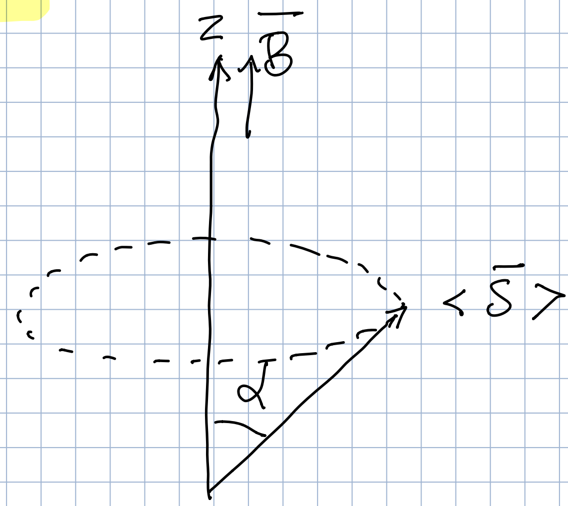
$$\langle S_y \rangle = -\frac{\hbar}{2} \sin\alpha \, \sin(\gamma B_0 t)$$

$$\langle S_z \rangle = \frac{\hbar}{2} \chi^{+}(t) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \chi(t)$$

$$= \frac{\hbar}{2} \left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} \right)$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos\alpha$$

$\langle \bar{S} \rangle$



LARMOR
PRECESSION

$$\omega_L = \gamma B_0$$

$\Longrightarrow$ e^- IN INHOMOGENES $\bar{B}$

$$-\bar{\mu}\cdot\bar{B}$$

$\bar{B}$ INHOMOGEN

$$\bar{F} = -\bar{\nabla} V$$

$$= \bar{\nabla}(\bar{\mu}\cdot\bar{B})$$

$\hookrightarrow$ $\bar{B} = B_0\,\bar{e}_z \rightsquigarrow \bar{F} = \bar{0}$

$\hookrightarrow$ $\bar{B} = -\alpha x\,\bar{e}_x + (\alpha z + B_0)\,\bar{e}_z$

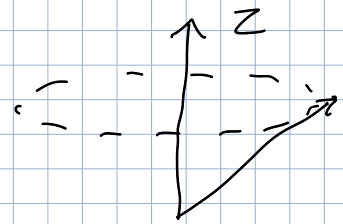
$\alpha \ll$

$$\bar{\mu}\cdot\bar{B} = -\alpha x\,\gamma S_x + (\alpha z + B_0)\,\gamma S_z$$

$$\bar{\mu} = \gamma\bar{S}$$

$$\rightsquigarrow \bar{F} = \alpha\gamma\left(-S_x\,\bar{e}_x + S_z\,\bar{e}_z\right)$$

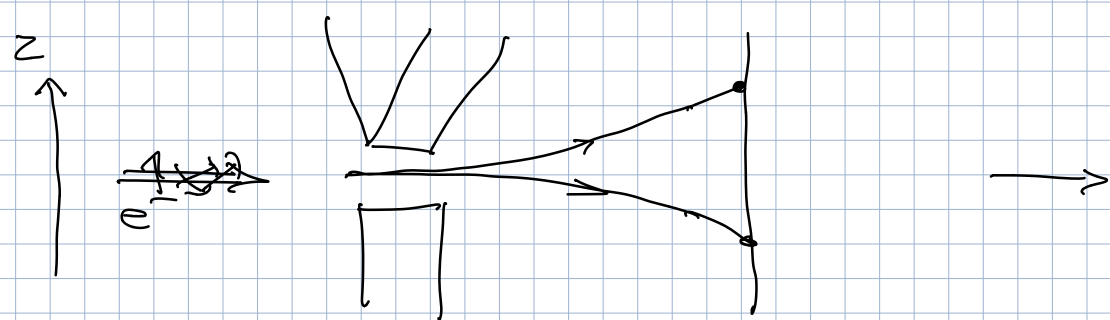
$\langle S_x \rangle$



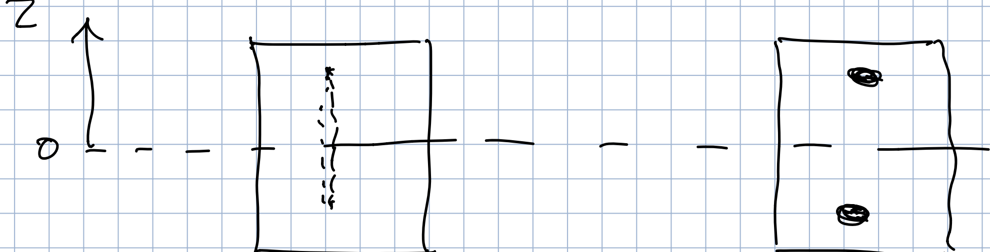
$\langle S_x \rangle = 0$

$$\rightsquigarrow \bar{F} = \alpha \gamma \underbrace{\langle S_z \rangle}_{\pm \hbar/2} \bar{e}_z$$

STERN GERLACH 1922



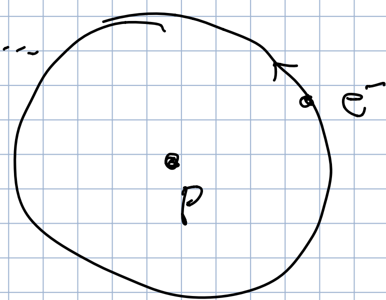
KLASSISCH QM



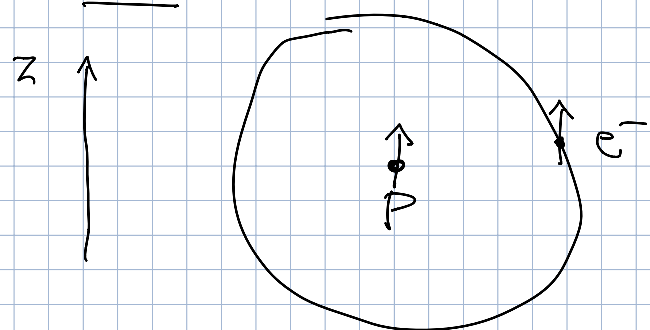
⇒ ADDITION VON 2 DREHIMPULSE

$\bar{L}$ $\quad L = 0, 1, 2, \ldots$

$\bar{S}$ $\quad S = \frac{1}{2}$



$n = 1 \quad \underline{L = 0}$



e^-, p $\quad$ SPIN 1/2

$|e^- p\rangle \qquad \uparrow\uparrow \qquad \uparrow\downarrow \qquad \downarrow\uparrow \qquad \downarrow\downarrow$

GESAMMT DREHIMPULS

$e^- \quad \bar{S}_1$

$p \quad \bar{S}_2$

$$\longrightarrow \quad \bar{S} = \bar{S}_1 + \bar{S}_2$$

$$S_{1z}\,\chi_1 = \hbar\, s_{1z}\,\chi_1 \qquad s_{1z} = \pm\frac{1}{2}$$

$$S_{2z}\,\chi_2 = \hbar\, s_{2z}\,\chi_2 \qquad s_{2z} = \pm\frac{1}{2}$$

$$|e^- p\rangle \longleftrightarrow \chi_1\,\chi_2$$

$$(S_{1z} + S_{2z})\,\chi_1\,\chi_2$$

$$= \hbar\,\underbrace{(s_{1z} + s_{2z})}_{\equiv\, s_z}\,\chi_1\,\chi_2$$

$$\uparrow\uparrow \qquad s_z = +1$$

$$\left.\begin{array}{l}\uparrow\downarrow\\ \downarrow\uparrow\end{array}\right\} \qquad s_z = 0$$

$$\downarrow\downarrow \qquad s_z = -1$$

$$\bar{S} = \bar{S}_1 + \bar{S}_2$$

$$J:\; 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \ldots \rightsquigarrow (2J+1)$$

$$S\,? \quad \begin{cases} \nearrow \; S = 1 \longrightarrow -1,\ 0,\ 1 \\ \searrow \; S = 0 \longrightarrow 0 \end{cases}$$

$S = 1$ $\qquad |S\, S_z\rangle$

SPIN TRIPLET

$$|1\ {+1}\rangle = |\uparrow\uparrow\rangle$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|1\ {-1}\rangle = |\downarrow\downarrow\rangle$$

$+1$ —— $|\uparrow\uparrow\rangle$

$\downarrow S_-$

0 ——

$\uparrow S_+$

-1 —— $|\downarrow\downarrow\rangle$

$S = 0$ $\qquad |0\ 0\rangle = \dfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

SPIN SINGLET

• e^- im $n = 2$, $\ell = 1$

e^- : $\ell = 1$, $s = \frac{1}{2}$

$\rightsquigarrow \bar{J} = \bar{L} + \bar{S}$

? J_z $(+1, 0, -1)$ $(\pm \frac{1}{2})$

$J = \frac{3}{2}$: $-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$ 4 $\longrightarrow 6$

$J = \frac{1}{2}$: $-\frac{1}{2}, \frac{1}{2}$ 2

• $\bar{J} = \bar{J}_1 + \bar{J}_2$

$J_z = J_{1z} + J_{1z}$

$|J_1 - J_2|, \ldots, J_1 + J_2$

$J : 0, \frac{1}{2}, 1, \frac{3}{2}$

$\rightarrow \quad J_1 = \frac{1}{2} \quad , \quad J_2 = \frac{1}{2}$

$J: \quad 0 \, , \, 1$

$\rightarrow \quad J_1 = 1 \quad , \quad J_2 = \frac{1}{2}$

$J: \quad \frac{1}{2} \, , \, \frac{3}{2}$

$\rightarrow \quad J_1 = 2 \quad , \quad J_2 = \frac{1}{2}$

$J: \quad \frac{3}{2} \, , \, \frac{5}{2}$

$\rightarrow \quad J_1 = 2 \quad , \quad J_2 = 3$

$J: \quad 1, \; 2, \; 3, \; 4, \; 5$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$3 \quad 5 \quad 7 \quad 9 \quad 11$

$$|JM\rangle = \sum_{\substack{m_1 \\ m_1 + m_2 = M}} \sum_{m_2} \underbrace{\langle j_1 m_1, j_2 m_2 | JM\rangle}_{\text{CLEBSCH-GORDAN COEFF.}} |j_1 m_1\rangle \, |j_2 m_2\rangle$$