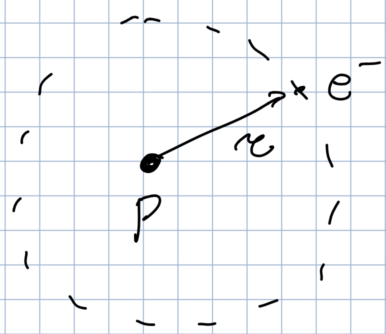
⟹ VORLESUNG 17 QM

H-ATOM



$$V(r) = -\frac{\alpha}{r}(\hbar c)$$

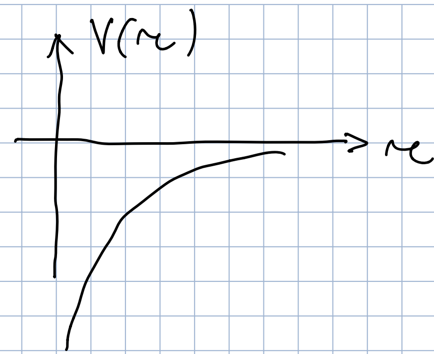
$$\alpha \simeq \frac{1}{137} \qquad (\hbar c \simeq 197 \text{ MeV fm})$$

$$\psi(\vec{r}) = R(r)\, Y_{lm}(\theta, \phi)$$

$$R(r) = \frac{u(r)}{r}$$

$$-\frac{\hbar^2}{2m}\frac{d^2 u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} \right] u = E\, u$$

$$V(r) = -\frac{\alpha}{r}\hbar c$$



GEBUNDENE ZUSTÄNDE

$$E = -\frac{\hbar^2}{2m} \kappa^2$$

$$\rho \equiv \kappa r$$

$$\frac{d^2 u}{d\rho^2} = \left[1 - \frac{\alpha \hbar c \, 2m}{\hbar^2 \kappa \rho} + \frac{\ell(\ell+1)}{\rho^2} \right] u$$

$\equiv \rho_0$

$$\rightsquigarrow \quad \frac{d^2 u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2} \right] u$$

↳ ASYMPTOTISCHE VERHALTEN

$$\rho \to \infty$$

$$\frac{d^2 u}{d\rho^2} = u$$

$$U(\rho) \underset{\rho \to \infty}{\Longrightarrow} \cancel{B e^{\rho}} + A e^{-\rho}$$

↳ e^- AM KERN

$$\rho \to 0$$

$$\frac{d^2 U}{d\rho^2} = \frac{\ell(\ell+1)}{\rho^2} U$$

$$U(\rho) \underset{\rho \to 0}{\longrightarrow} C \rho^{\ell+1} + \cancel{D \rho^{-\ell}}$$

$$U(\rho) = \rho^{a}$$

$$a(a-1)\rho^{a-2} = \ell(\ell+1)\rho^{a-2}$$

$$a(a-1) = \ell(\ell+1)$$

$$\nearrow a = \ell + 1$$

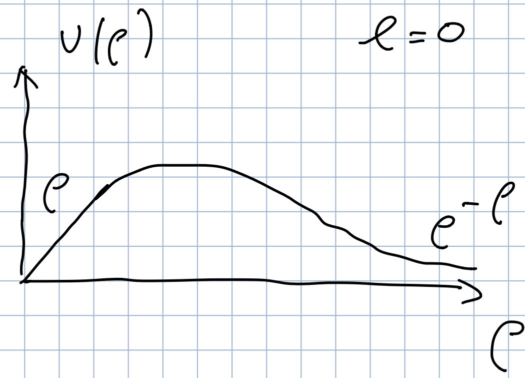
$$\searrow a = -\ell$$

$$R(r) = \frac{U(r)}{r} = K \frac{U(\rho)}{\rho}$$

$$u(\rho) \xrightarrow[\rho \to 0]{} 0$$

$$\ell \in \mathbb{N} \qquad \ell = 0, 1, 2, \ldots$$

$$\ell = 0$$



$$\rightsquigarrow \quad u(\rho) = \rho^{\ell+1} e^{-\rho} \, v(\rho)$$

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$$

$$\frac{du}{d\rho} = e^{-\rho} \left[(\ell+1) \rho^{\ell} v - \rho^{\ell+1} v + \rho^{\ell+1} \frac{dv}{d\rho} \right]$$

$$= \rho^{\ell} e^{-\rho} \left[(\ell+1-\rho) v + \rho \frac{dv}{d\rho} \right]$$

$$\frac{d^2u}{d\rho^2} = e^{-\rho} \Big[-(\ell+1-\rho) \rho^{\ell} v - \rho^{\ell+1} \frac{dv}{d\rho}$$

$$+ \ell(\ell+1) \rho^{\ell-1} v$$

$$- (\ell+1) \rho^{\ell} v$$

$$+ (\ell+1-\rho)\,\rho^{\ell}\,\frac{dv}{d\rho}$$

$$+ (\ell+1)\,\rho^{\ell}\,\frac{dv}{d\rho}$$

$$+ \rho^{\ell+1}\,\frac{d^2v}{d\rho^2}\Bigg]$$

$$\frac{d^2u}{d\rho^2} = \underbrace{\rho^{\ell} e^{-\rho}}\Bigg[\left(-2(\ell+1) + \rho + \frac{\ell(\ell+1)}{\rho}\right)v$$

$$+ \left(2(\ell+1) - 2\rho\right)\frac{dv}{d\rho}$$

$$+ \rho\,\frac{d^2v}{d\rho^2}\Bigg]$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u$$

$$\rho\,\frac{d^2v}{d\rho^2} + 2(\ell+1-\rho)\,\frac{dv}{d\rho}$$

$$+ \left(-2(\ell+1) + \cancel{\rho} + \cancel{\frac{\ell(\ell+1)}{\rho}}\right)v$$

$$= \rho v\left[\cancel{1} - \frac{\rho_0}{\rho} + \cancel{\frac{\ell(\ell+1)}{\rho^2}}\right]$$

$$\rho \frac{d^2 v}{d\rho^2} + 2(\ell+1-\rho)\frac{dv}{d\rho} + \left[\rho_0 - 2(\ell+1)\right] v = 0$$

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$$

$$\frac{dv}{d\rho} = \sum_{j=1}^{\infty} j\, c_j\, \rho^{j-1} = \sum_{j=0}^{\infty} (j+1) c_{j+1} \rho^j$$

$$\frac{d^2 v}{d\rho^2} = \sum_{j=\cancel{1}\,0}^{\infty} j(j+1)\, c_{j+1}\, \rho^{j-1}$$

TERME $\sim \rho^j$

$$\sum_{j=0}^{\infty} j(j+1)\, c_{j+1}\, \rho^{j}$$

$$+ \; 2(\ell+1-\rho) \sum_{j=0}^{\infty} (j+1)\, c_{j+1}\, \rho^{j}$$

$$+ \; \left[\rho_0 - 2(\ell+1)\right] \sum_{j=0}^{\infty} c_j\, \rho^{j} = 0$$

$$\rho \sum_{j=0}^{\infty} (j+1)\, c_{j+1}\, \rho^{j}$$

$$= \sum_{j=0}^{\infty} j\, c_j\, \rho^{j}$$

$$\sum_{j=0}^{\infty} \left[\cdots\right] \rho^{j} = 0$$

$$j(j+1)\, c_{j+1} + 2(\ell+1)(j+1)\, c_{j+1}$$

$$- 2j\, c_j + \left[\rho_0 - 2(\ell+1)\right] c_j = 0$$

$$c_{j+1} = \frac{\left[2(j+l+1) - \rho_0\right]}{(j+1)\,(j+2l+2)}\, c_j$$

$$v(\rho) = \rho^{l+1} e^{-\rho} \sum_{j=0}^{\infty} c_j \rho^j$$

$j >>$

$$c_{j+1} \simeq \frac{2j}{(j+1)\,j}\, c_j = \frac{2}{j+1}\, c_j$$

$$c_1 = 2\, c_0$$

$$c_2 = \frac{2}{2}\, c_1 = \frac{2^2}{2}\, c_0$$

$$c_3 = \frac{2}{3}\, c_2 = \frac{2^3}{3!}\, c_0$$

$$\vdots$$

$j >>$ $$c_j = \frac{2^j}{j!}\, c_0$$

$$v(\rho) \xrightarrow[\rho \to \infty]{} \sum_{j=0}^{\infty} \frac{2^j}{j!} \rho^j = \sum_{j=0}^{\infty} \frac{(2\rho)^j}{j!}$$

$$= e^{2\rho}$$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

$$\underset{\rho \to \infty}{\longrightarrow} \rho^{l+1} e^{-\rho} e^{2\rho} \sim \cancel{\rho^{l+1} e^{\rho}}$$

REIHE MUSS ABBRECHEN

j_{max}

$$v(\rho) = \sum_{j=0}^{j_{max}} c_j \rho^j$$

$$c_{j_{max}+1} = 0$$

$$2(j_{max} + l + 1) = \rho_0$$

$$j_{max} \geq 0$$

$$l \geq 0$$

$$n \equiv j_{max} + l + 1$$

$$n = 1, 2, \ldots$$

HAUPTQUANTENZAHL

$$\rho_0 = \frac{\alpha \hbar c \, 2m}{\hbar^2 K} = 2n \qquad E = -\frac{\hbar^2 K^2}{2m}$$

$$K = \frac{2m}{\hbar^2} \, \frac{\alpha \hbar c}{2n}$$

$$E = -\left(\frac{2m}{\hbar^2}\right) \frac{\alpha^2 (\hbar c)^2}{4 \, n^2}$$

$$E = -\left(\frac{1}{2} m \alpha^2 c^2\right) \frac{1}{n^2}$$

$$E_n = -\frac{E_1}{n^2}$$

BOHR FORMEL

$$n = 1, 2, \ldots$$

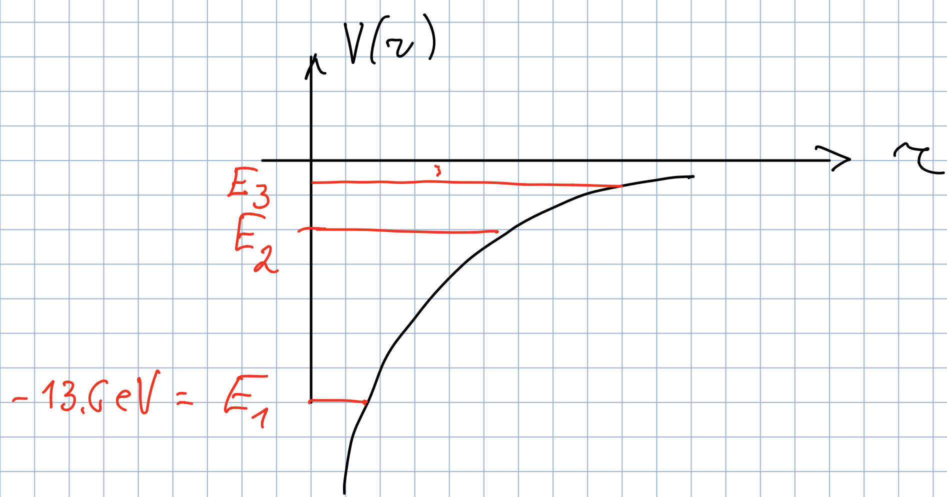
GRUNDZUSTAND

$$E_1 \equiv \left(\frac{1}{2} m c^2\right) \alpha^2$$

$$\simeq \frac{1}{2} \, 0.5 \, \text{MeV} \left(\frac{1}{137}\right)^2$$

$$= 0.25 \, \frac{10^6}{(1.37)^2 \, 10^4} \quad eV$$

$$= \frac{25}{(1.37)^2} \, eV \simeq 13{,}6 \, eV$$



$$U(\rho) \sim e^{-\rho} = e^{-\kappa r}$$

$$\kappa = \frac{2m}{\hbar^2} \frac{\alpha \hbar c}{2n} \equiv \frac{1}{a} \frac{1}{n}$$

↑ LÄNGE

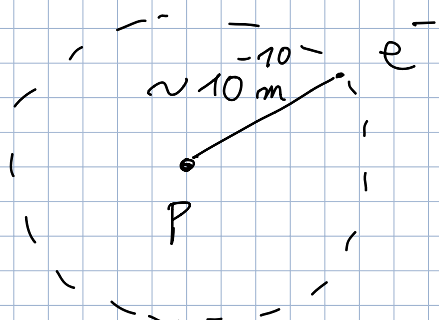
BOHR RADIUS $\Rightarrow$ $a = \frac{1 \quad \hbar c}{\alpha m \quad c^2}$

$$a = \frac{(137)\ (197\ \text{MeV fm})}{(0.5\ \text{MeV})}$$

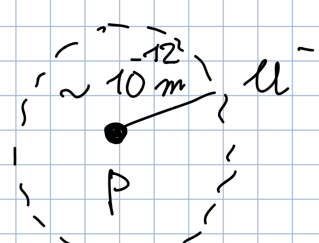
$$\simeq 1.37\ 10^{2}\ 4\ 10^{2}\ 10^{-15}\ m$$

$$a = 0.53\ 10^{-10}\ m$$

$$U(\rho) \xrightarrow[\rho >]{} e^{-\frac{r}{a n}} \underset{n=1}{\sim} e^{-\frac{r}{a}}$$



MUONISCHE ATOME



$$m_\mu = 105\ \text{MeV}$$

$$\simeq 200\ m_e$$

$\rightarrow$ 200 × KLEINER