

⇒ VORLESUNG 15 QM

QM IN 3 DIM

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}, t) \right) \Psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \Psi$$

$$\vec{r} (x, y, z)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

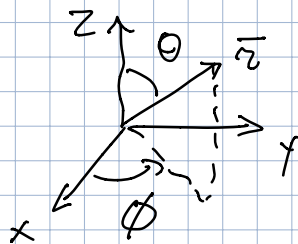
$$V = V(\vec{r})$$

$$\Psi(\vec{r}, t) = \psi(\vec{r}) e^{-\frac{i}{\hbar} Et}$$

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) \right) \psi(\vec{r}) = E \psi(\vec{r})$$

ZENTRALE POT.

$$\hookrightarrow V(\vec{r}) = V(r)$$



$$r = |\vec{r}|$$

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$$\Psi(\vec{r}) = R(r) Y(\theta, \phi)$$

$$\underbrace{2 \text{ Gl}} \begin{array}{l} \nearrow r \\ \searrow \theta, \phi \end{array} \quad \underline{V(r)}$$

$$Y(\theta, \phi) = C e^{im\phi} P_e^m(\cos\theta)$$

l, m

DREHIMPULS

• KLASSISCH

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} (L_x, L_y, L_z)$$

$$\left\{ \begin{array}{l} L_x = y p_z - z p_y \\ L_y = z p_x - x p_z \\ L_z = x p_y - y p_x \end{array} \right.$$

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• QM $\vec{p} \rightarrow \hat{p} = -i\hbar \vec{\nabla}$

$$\hat{L} = -i\hbar \vec{r} \times \vec{\nabla}$$

$$\hat{L}_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

→ EIGENFUNKTIONEN / EIGENWERTE

$$[\hat{L}_x, \hat{L}_y] = [y \hat{p}_z - z \hat{p}_y, z \hat{p}_x - x \hat{p}_z]$$

$$= [y \hat{p}_z, z \hat{p}_x]$$

$$- [z \hat{p}_y, z \hat{p}_x]$$

$$- [y \hat{p}_z, x \hat{p}_z]$$

$$+ [z \hat{p}_y, x \hat{p}_z]$$

$$= y \hat{p}_x [\hat{p}_z, z] + 0 + 0$$

$$+ \hat{p}_y \times [z, \hat{p}_z]$$

$$\begin{array}{c} \curvearrowright \\ \times \hat{p}_y \end{array}$$

$$\downarrow [z, \hat{p}_z] = i\hbar$$

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$$= i\hbar \underbrace{(x \hat{p}_y - y \hat{p}_x)}_{\hat{L}_z}$$

$$\begin{cases} [\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z \\ [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \\ [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y \end{cases}$$

NICHT - KOMMUTIEREN

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

$$\hat{A} = \hat{L}_x$$

$$\hat{B} = \hat{L}_y$$

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i} \langle [\hat{L}_x, \hat{L}_y] \rangle \right)^2$$

$i\hbar \hat{L}_z$

$$\geq \frac{\hbar^2}{4} \langle \hat{L}_z \rangle^2$$

KEINE GEMEINSAMEN EIGENZUSTÄNDE

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$$\rightarrow L^2 = L_x^2 + L_y^2 + L_z^2$$

$$[L^2, L_x] = [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$$

$$\downarrow \rightarrow [AB, C] = A[B, C] + [A, C]B$$

$$ABC - CAB = A(BC - CB) + (AC - CA)B \\ \stackrel{!}{=} ABC - CAB$$

$$[L^2, L_x] = L_y [L_y, L_x] + [L_y, L_x] L_y + L_z [L_z, L_x] + [L_z, L_x] L_z \\ = -i\hbar L_y L_z - i\hbar L_z L_y + i\hbar L_z L_y + i\hbar L_y L_z \\ = 0$$

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$$\begin{cases} [L^2, L_x] = 0 \\ [L^2, L_y] = 0 \\ [L^2, L_z] = 0 \end{cases}$$

L^2, L_z GEMEINSAME EIGENFUNKT.

⇒ LEITER OPERATOREN

$$\hat{L}_{\pm} = \hat{L}_x \pm i \hat{L}_y$$

$$\begin{aligned} [\hat{L}_z, \hat{L}_{\pm}] &= [\hat{L}_z, \hat{L}_x] \pm i [\hat{L}_z, \hat{L}_y] \\ &= \underbrace{[\hat{L}_z, \hat{L}_x]}_{i\hbar \hat{L}_y} \pm i \underbrace{[\hat{L}_z, \hat{L}_y]}_{-i\hbar \hat{L}_x} \\ &= \pm \hbar (\hat{L}_x \pm i \hat{L}_y) \end{aligned}$$

$$\begin{cases} [\hat{L}_z, \hat{L}_{\pm}] = \pm \hbar \hat{L}_{\pm} \\ [\hat{L}^2, \hat{L}_{\pm}] = 0 \end{cases}$$

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$$\begin{cases} L^2 f = \lambda f \\ L_z f = \mu f \end{cases} \quad \lambda, \mu \in \mathbb{R}$$

$(L_{\pm} f)$ EIGENFUNKTION L^2, L_z

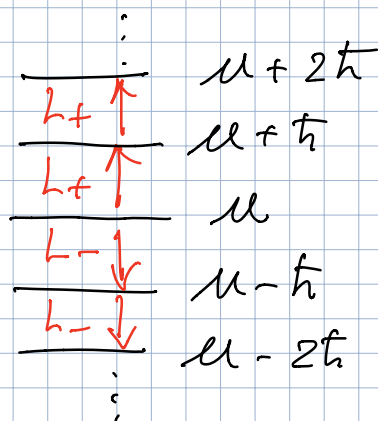
$$\hookrightarrow L^2 (\underline{L_{\pm} f}) = L_{\pm} L^2 f = \lambda (\underline{L_{\pm} f})$$

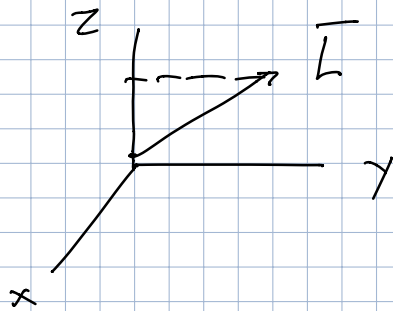
↑
SELBEN
EIGENWERT

$$\begin{aligned} \hookrightarrow L_z (\underline{L_{\pm} f}) &= L_{\pm} \underbrace{L_z f}_{\mu f} \pm \hbar \underbrace{L_{\pm} f}_{\text{}} \\ &= (\mu \pm \hbar) (\underline{L_{\pm} f}) \end{aligned}$$

EIGENWERT

$$\mu \pm \hbar$$





MAX WERT VON L_z

$$L_z f_{\text{top}} \equiv \underbrace{\hbar l}_{\mu_{\text{max}}} f_{\text{top}} \quad \begin{matrix} z \uparrow \\ \uparrow \bar{L} \end{matrix}$$

$$L_+ f_{\text{top}} = 0$$

$$\bar{l} \leq l \quad L_z f_{\text{bottom}} \equiv \underbrace{\hbar \bar{l}} f_{\text{bottom}}$$

$$L_- f_{\text{bottom}} = 0$$

\hookrightarrow IDENTITÄT

$$\begin{aligned} L_+ L_- &= (L_x + i L_y)(L_x - i L_y) \\ &= L_x^2 + L_y^2 \mp i (L_x L_y - L_y L_x) \\ &\quad \underbrace{\hspace{10em}}_{i \hbar L_z} \end{aligned}$$

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$$= L_x^2 + L_y^2 \pm \hbar L_z$$

$$L_{\pm} L_{\mp} = L^2 - L_z^2 \pm \hbar L_z$$

$$\rightsquigarrow L^2 = L_{\pm} L_{\mp} + L_z^2 \mp \hbar L_z$$

$$\begin{aligned} L^2 \psi_{\text{top}} &= (\cancel{L_- L_+} + L_z^2 + \hbar L_z) \psi_{\text{top}} \\ &= (\hbar^2 l^2 + \hbar(\hbar l)) \psi_{\text{top}} \\ &= \hbar^2 l(l+1) \psi_{\text{top}} \\ &\quad \underbrace{\hspace{10em}}_{\lambda} \end{aligned}$$

$$\lambda = \hbar^2 l(l+1)$$

$$\begin{aligned} L^2 \psi_{\text{bottom}} &= (\cancel{L_+ L_-} + L_z^2 - \hbar L_z) \psi_{\text{bottom}} \\ &= (\hbar^2 \bar{l}^2 - \hbar^2 \bar{l}) \psi_{\text{bottom}} \\ &= \hbar^2 \bar{l}(\bar{l}-1) \psi_{\text{bottom}} \\ &\quad \underbrace{\hspace{10em}}_{\lambda} \end{aligned}$$

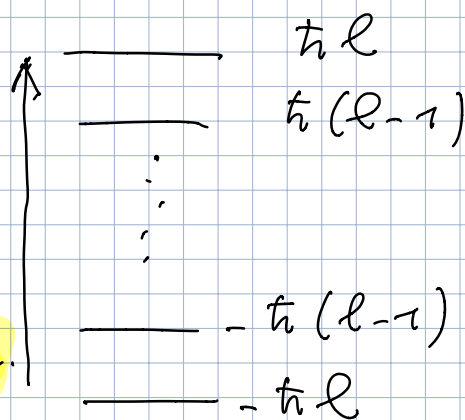
$$l(l+1) = \bar{l}(\bar{l}-1)$$

$$\rightarrow \bar{l} = -l$$

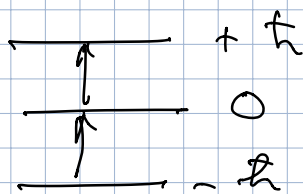
$$\rightarrow \bar{l} = \cancel{l+1}$$

$$2l \in \mathbb{N}$$

$$l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$$



$$l=1$$



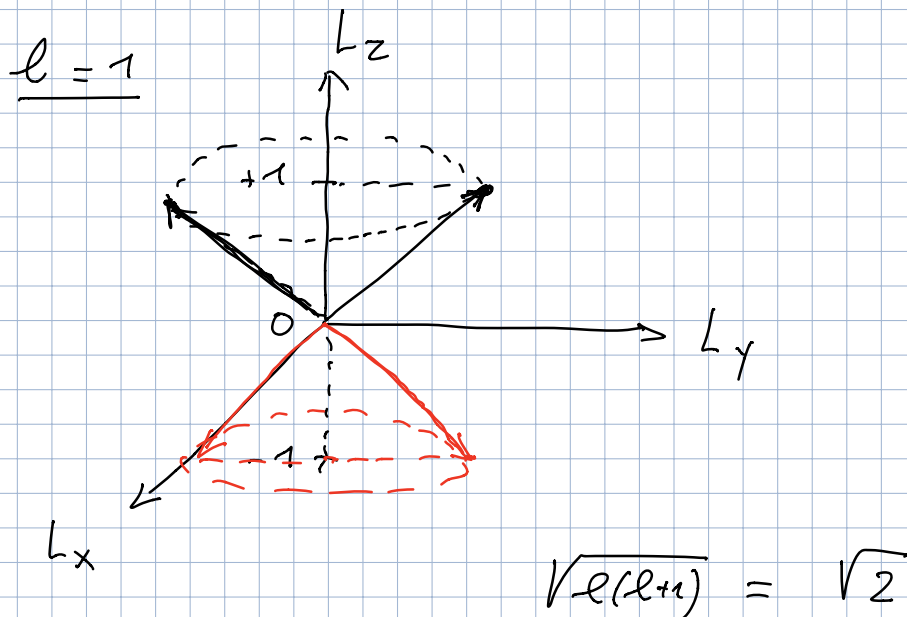
$$\hat{L}_z f \equiv \underbrace{\hbar m}_{l} f$$

$$m = -l, \dots, +l$$

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$$\begin{cases} L^2 \psi_{lm} = \hbar^2 \underline{l(l+1)} \psi_{lm} \\ L_z \psi_{lm} = \hbar \underline{m} \psi_{lm} \end{cases}$$

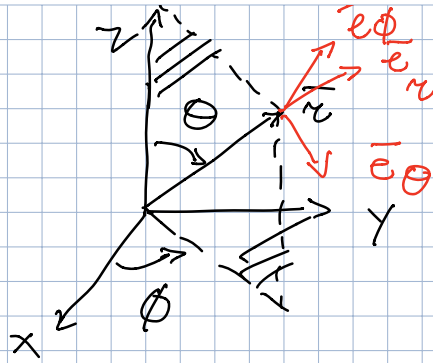
$$\begin{cases} l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \\ m = -l, \dots, +l \end{cases}$$



$$\vec{L} = -i\hbar (\vec{r} \times \vec{\nabla})$$

$$\vec{r} = r \vec{e}_r$$

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$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

$$\vec{r} = r \vec{e}_r$$

$$\vec{e}_r \times \vec{e}_\theta = \vec{e}_\phi$$

$$\vec{e}_r \times \vec{e}_\phi = -\vec{e}_\theta$$

$$\vec{L} = -i\hbar \left(\vec{e}_\phi \frac{\partial}{\partial \theta} - \vec{e}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right)$$

NUR θ, ϕ ABHÄNGIGKEIT!

$$L_z, L^2$$

$$1) \vec{e}_\phi, \vec{e}_\theta \rightarrow \vec{e}_x, \vec{e}_y, \vec{e}_z$$

$$L_x, L_y, L_z$$

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$$2) \quad L^2 = L_x^2 + L_y^2 + L_z^2$$

$$3) \quad L_z \psi_{lm} = \hbar m \psi_{lm}$$

$$L^2 \psi_{lm} = \hbar^2 l(l+1) \psi_{lm}$$

$$\begin{aligned} \psi_{lm}(\theta, \phi) &= Y_{lm}(\theta, \phi) \\ &= C e^{im\phi} P_l^m(\cos\theta) \end{aligned}$$

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + V(r) \psi(\vec{r}) = E \psi(\vec{r})$$

$$\psi(\vec{r}) = R(r) Y_{lm}(\theta, \phi)$$

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