$\Longrightarrow$ VORLESUNG 15 QM

QM IN 3 DIM

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\bar{r},t) \right) \Psi(\bar{r},t) = i\hbar \frac{\partial}{\partial t} \Psi$$

$$\bar{r}\ (x, y, z)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

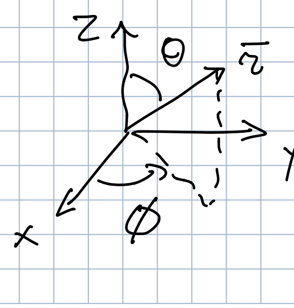
$$V = V(\bar{r})$$

$$\Psi(\bar{r},t) = \psi(\bar{r})\, e^{-\frac{i}{\hbar}Et}$$

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\bar{r}) \right) \psi(\bar{r}) = E\, \psi(\bar{r})$$

ZENTRALE POT.

$$\hookrightarrow V(\bar{r}) = V(r)$$



$$r = |\bar{r}|$$

$$\psi(\bar{r}) = R(r)\, Y(\theta, \phi)$$

2 GL → r $V(r)$
→ θ, ϕ

$$Y(\theta, \phi) = C\, e^{im\phi}\, P_l^m(\cos\theta)$$

l, m

DREHIMPULS

- KLASSISCH

$$\bar{L} = \bar{r} \times \bar{p}$$

$$\bar{L}(L_x, L_y, L_z)$$

$$\begin{cases} L_x = y p_z - z p_y \\ L_y = z p_x - x p_z \\ L_z = x p_y - y p_x \end{cases}$$

• QM

$$\bar{p} \longrightarrow \hat{\bar{p}} = -i\hbar \bar{\nabla}$$

$$\hat{\bar{L}} = -i\hbar \, \bar{r} \times \bar{\nabla}$$

$$\hat{L}_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

→ EIGENFUNKTIONEN / EIGENWERTE

$$[\hat{L}_x, \hat{L}_y] = [y\hat{p}_z - z\hat{p}_y, \; z\hat{p}_x - x\hat{p}_z]$$

$$= [y\hat{p}_z, z\hat{p}_x]$$

$$- [z\hat{p}_y, z\hat{p}_x]$$

$$- [y\hat{p}_z, x\hat{p}_z]$$

$$+ [z\hat{p}_y, x\hat{p}_z]$$

$$= y\hat{p}_x [\hat{p}_z, z] + 0 + 0$$

$$+ \hat{p}_y x [z, \hat{p}_z]$$

↷ $x\hat{p}_y$

↓ $[z, \hat{p}_z] = i\hbar$

$$= i\hbar \underbrace{\left(x \hat{p}_y - y \hat{p}_x \right)}_{\hat{L}_z}$$

$$\begin{cases} [\hat{L}_x , \hat{L}_y] = i\hbar \hat{L}_z \\ [\hat{L}_y , \hat{L}_z] = i\hbar \hat{L}_x \\ [\hat{L}_z , \hat{L}_x] = i\hbar \hat{L}_y \end{cases}$$

NICHT - KOMMUTIEREN

$$\sigma_A^2 \ \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

$$\hat{A} = \hat{L}_x$$

$$\hat{B} = \hat{L}_y$$

$$\sigma_{L_x}^2 \ \sigma_{L_y}^2 \geq \left(\frac{1}{2i} \langle \underbrace{[\hat{L}_x , \hat{L}_y]}_{i\hbar \hat{L}_z} \rangle \right)^2$$

$$\geq \frac{\hbar^2}{4} \langle \hat{L}_z \rangle^2$$

KEINE GEMEINSAMEN EIGENZUSTÄNDE

$$\rightarrow \quad L^2 = L_x^2 + L_y^2 + L_z^2$$

$$[L^2, L_x] = [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$$

$$\downarrow \quad \rightarrow [AB, C] = A[B, C] + [A, C]B$$

$$ABC - CAB = A(BC - CB) + (AC - CA)B \overset{!}{=} ABC - CAB$$

$$[L^2, L_x] = L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$$

$$= -i\hbar L_y L_z - i\hbar L_z L_y + i\hbar L_z L_y + i\hbar L_y L_z$$

$$= 0$$

$$\begin{cases} [L^2, L_x] = 0 \\ [L^2, L_y] = 0 \\ [L^2, L_z] = 0 \end{cases}$$

L^2, L_z GEMEINSAME EIGENFUNKT.

$\Rightarrow$ LEITER OPERATOREN

$$\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y$$

$$[\hat{L}_z, \hat{L}_\pm] = \underbrace{[\hat{L}_z, \hat{L}_x]}_{i\hbar \hat{L}_y} \pm i \underbrace{[\hat{L}_z, \hat{L}_y]}_{-i\hbar \hat{L}_x}$$

$$= \pm \hbar \left(\hat{L}_x \pm i \hat{L}_y \right)$$

$$\begin{cases} [\hat{L}_z, \hat{L}_\pm] = \pm \hbar \hat{L}_\pm \\ [\hat{L}^2, \hat{L}_\pm] = 0 \end{cases}$$

$$\begin{cases} L^2 f = \lambda f \\ L_z f = \mu f \end{cases} \qquad \lambda, \mu \in \mathbb{R}$$

$(L_\pm f)$ EIGENFUNKTION L^2, L_z

$\hookrightarrow \quad L^2 (\underline{\underline{L_\pm f}}) = L_\pm L^2 f = \lambda (\underline{\underline{L_\pm f}})$

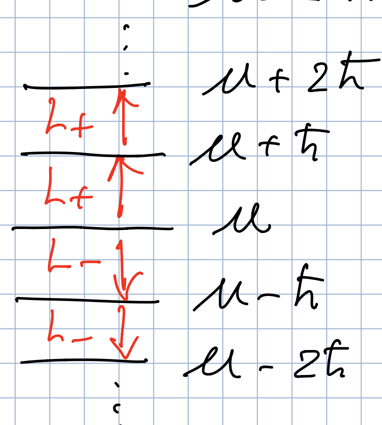
↑ SELBEN EIGENWERT

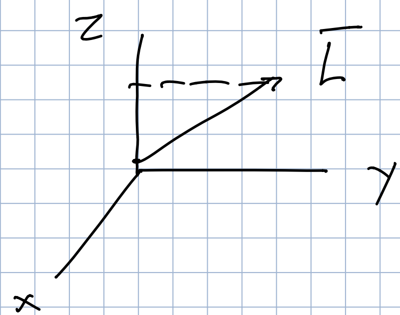
$\hookrightarrow \quad L_z (\underline{\underline{L_\pm f}}) = L_\pm \underbrace{L_z f}_{\mu f} \pm \hbar L_\pm f$

$$= (\mu \pm \hbar)(\underline{\underline{L_\pm f}})$$

EIGENWERT

$\mu \pm \hbar$





MAX WERT VON L_z

$$L_z f_{top} = \hbar \underbrace{l}_{m_{max}} f_{top} \qquad z \uparrow \uparrow \vec{L}$$

$$L_+ f_{top} = 0$$

$\bar{l} \leq l$ $$L_z f_{bottom} = \hbar \underbrace{\bar{l}} f_{bottom}$$

$$L_- f_{bottom} = 0$$

↳ IDENTITÄT

$$L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y)$$

$$= L_x^2 + L_y^2 \mp i \underbrace{(L_x L_y - L_y L_x)}_{i\hbar L_z}$$

$$= L_x^2 + L_y^2 \pm \hbar L_z$$

$$L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z$$

$$\rightsquigarrow \quad L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

$$L^2 f_{top} = \left(\cancel{L_- L_+} + L_z^2 + \hbar L_z \right) f_{top}$$

$$= \left(\hbar^2 \ell^2 + \hbar (\hbar \ell) \right) f_{top}$$

$$= \underbrace{\hbar^2 \ell (\ell + 1)}_{\lambda} f_{top}$$

$$\lambda = \hbar^2 \ell (\ell + 1)$$

$$L^2 f_{bottom} = \left(\cancel{L_+ L_-} + L_z^2 - \hbar L_z \right) f_{bottom}$$

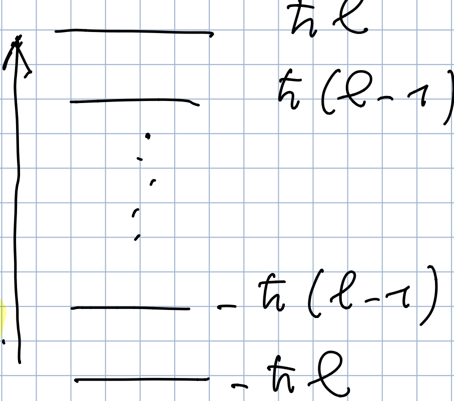
$$= \left(\hbar^2 \bar{\ell}^2 - \hbar^2 \bar{\ell} \right) f_{bottom}$$

$$= \underbrace{\hbar^2 \bar{\ell} (\bar{\ell} - 1)}_{\lambda} f_{bottom}$$

$$l(l+1) = \bar{l}(\bar{l}-1)$$

$\rightarrow \bar{l} = -l$

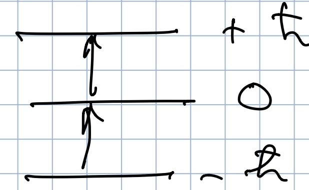
$\rightarrow$ ~~$\bar{l} = l + 1$~~



$2l \in \mathbb{N}$

$l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$l = 1$



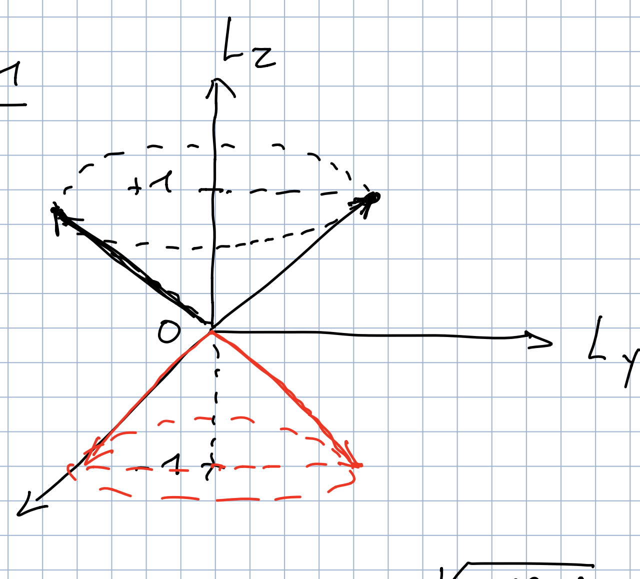
$$\hat{L}_z f = \underbrace{\hbar m}_{\mu} f$$

$m = -l, \ldots, +l$

$$\begin{cases} L^2 \, f_{lm} = \hbar^2 \, \underline{\underline{l(l+1)}} \, f_{lm} \\ L_z \, f_{lm} = \hbar \, \underline{\underline{m}} \, f_{lm} \end{cases}$$

$$\begin{cases} l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \\ m = -l, \ldots, +l \end{cases}$$

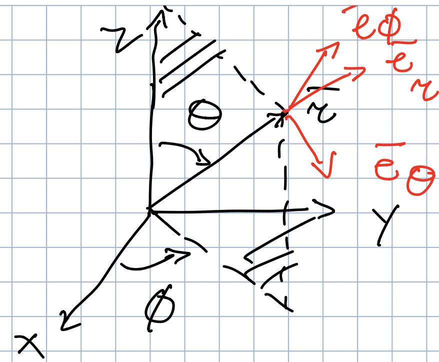
$\underline{l = 1}$



$$\sqrt{l(l+1)} = \sqrt{2}$$

$$\bar{L} = -i\hbar \left(\bar{r} \times \bar{\nabla} \right)$$

$$\bar{r} = r \, \bar{e}_r$$



$$\bar{\nabla} = \bar{e}_r \frac{\partial}{\partial r} + \bar{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \bar{e}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}$$

$$\bar{r} = r \, \bar{e}_r$$

$$\bar{e}_r \times \bar{e}_\theta = \bar{e}_\phi$$

$$\bar{e}_r \times \bar{e}_\phi = - \bar{e}_\theta$$

$$\bar{L} = -i\hbar \left(\bar{e}_\phi \frac{\partial}{\partial \theta} - \bar{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial \phi} \right)$$

NUR θ, ϕ ABHÄNGIGKEIT !

L_z, L^2

1) $\bar{e}_\phi, \bar{e}_\theta \rightarrow \bar{e}_x, \bar{e}_y, \bar{e}_z$

L_x, L_y, L_z

2) $L^2 = L_x^2 + L_y^2 + L_z^2$

3) $L_z f_{lm} = \hbar m f_{lm}$

$L^2 f_{lm} = \hbar l(l+1) f_{lm}$

$$f_{lm}(\theta,\phi) = Y_{lm}(\theta,\phi)$$

$$= C e^{im\phi} P_l^m(\cos\theta)$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\bar{r}) + V(r)\psi(\bar{r}) = E\psi(\bar{r})$$

$$\psi(\bar{r}) = R(r)\, Y_{lm}(\theta,\phi)$$