Lecture 22

Last lecture: we took a specific process (Moller scatt.)

Methods: trace manipulations

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m$$

$$\gamma^\mu\gamma^\nu\gamma^\alpha = S^{\mu\nu\alpha\beta}\gamma_\beta + i\varepsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\beta$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$$

$$\vdots$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$$

$$\mathrm{Tr}\,\{\gamma^{\alpha_1}\dots\gamma^{\alpha_{2n+1}}\} = 0$$

$$\mathrm{Tr}\,\gamma_5\{\gamma^{\alpha_1}\dots\gamma^{\alpha_{2n+1}}\} = 0$$

$$\mathrm{Tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 4i\varepsilon^{\mu\nu\alpha\beta}$$

$$\downarrow$$

$$\rightarrow \overline{|\mathcal{M}|^2} = M^2(s,t,u,m^2\dots)$$

$\downarrow$ pick a reference frame

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}(E,\theta)$$

Helicity formalism

Start w. invariant ampl.

$$\mathcal{M}(s,t,u;h_1,h_2,h_3,h_4)$$

$$\big|$$

↓ pick a reference frame

pick a representation

(Weyl, Dirac, Majorana)

Weyl → ultra relativistic

Dirac → non-relativistic

↓

Calculate helicity amplitudes

$$\mathcal{M}_{h_4 h_3; h_2 h_1}$$

↓

$$\frac{1}{4} \sum_{h_{1,2,3,4}} |\mathcal{M}_{h_4 h_3; h_2 h_1}|^2$$

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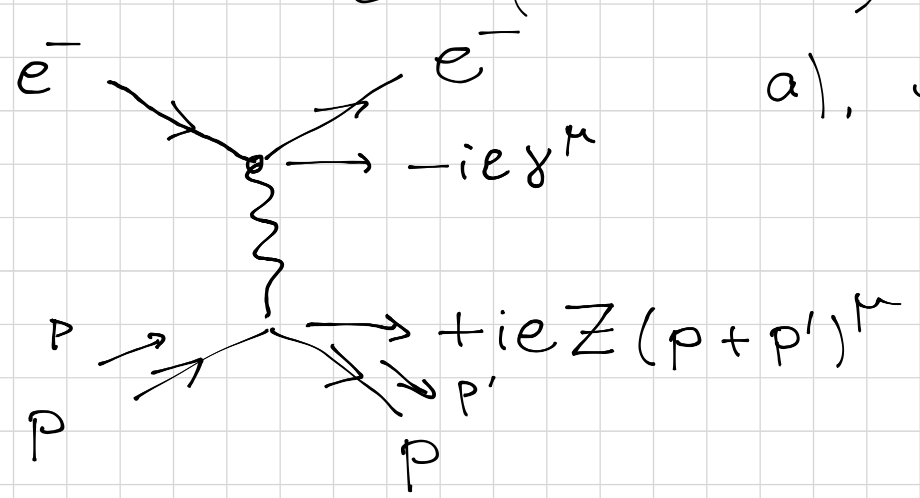
e – nucleon (nucleus) scattering



a). Spin-0 proton

$$\mathcal{M}^a = -\frac{Ze^2}{t} \bar{u}(k')(\not{p}+\not{p}')u(k)$$

b). Spin-½

$$\mathcal{M}^b = -\frac{Ze^2}{t} \bar{u}(k')\gamma^\mu u(k)\, \bar{u}(p')\gamma_\mu u(p)$$

$$|\overline{\mathcal{M}^a}|^2 = \frac{Z^2 e^4}{t^2} \cdot l^{\mu\nu} \cdot h^a_{\mu\nu}$$

$$= \frac{Z^2 e^4}{t^2} \sum_s \left(\bar{u}(k')\gamma^\mu u(k)\right)^\dagger \bar{u}(k')\gamma^\nu u(k)\, (p+p')_\mu (p+p')_\nu$$

$$l^{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\left[(\not{k}'+m)\gamma^{\mu}(\not{k}+m)\gamma^{\nu}\right]$$
$$= 2\left[k'^{\mu}k^{\nu} + k'^{\nu}k^{\mu} + \frac{t}{2}g^{\mu\nu}\right]$$

$$h^{a}_{\mu\nu} = (p+p')_{\mu}\,(p+p')_{\nu}$$

$$l^{\mu\nu}\cdot h^{a}_{\mu\nu} = 4(p+p',k)(p+p',k') + t(p+p')^2$$

Mandelstam variables

$$s+t+u = 2M^2 + \cancel{2m^2}\to 0 \qquad m \ll E \ll M$$

"Crossing variable" $\nu = \frac{s-u}{4M}$ $\qquad \frac{t}{4M^2} \ll 1$

$$2M\nu = \frac{1}{2}(p+p', k+k') = \underline{s - M^2 + t/2}$$

$$t = (p-p')^2$$

$$(p+p', k) = (p+p', k')$$

$$\hookrightarrow l^{\mu\nu}h^{a}_{\mu\nu} = (4M\nu)^2 + t(4M^2 - t)$$
$$= \left[(s-M^2)^2 + st\right]4$$

Use c.m. frame $\qquad s - M^2 = 2\sqrt{s}\,E_e$

$$t = -4E_e^2 \sin^2\frac{\theta}{2}$$

$$\downarrow\; l^{\mu\nu}h^{a}_{\mu\nu} = 16\,s\,E_e^2\cos^2\frac{\theta}{2}$$

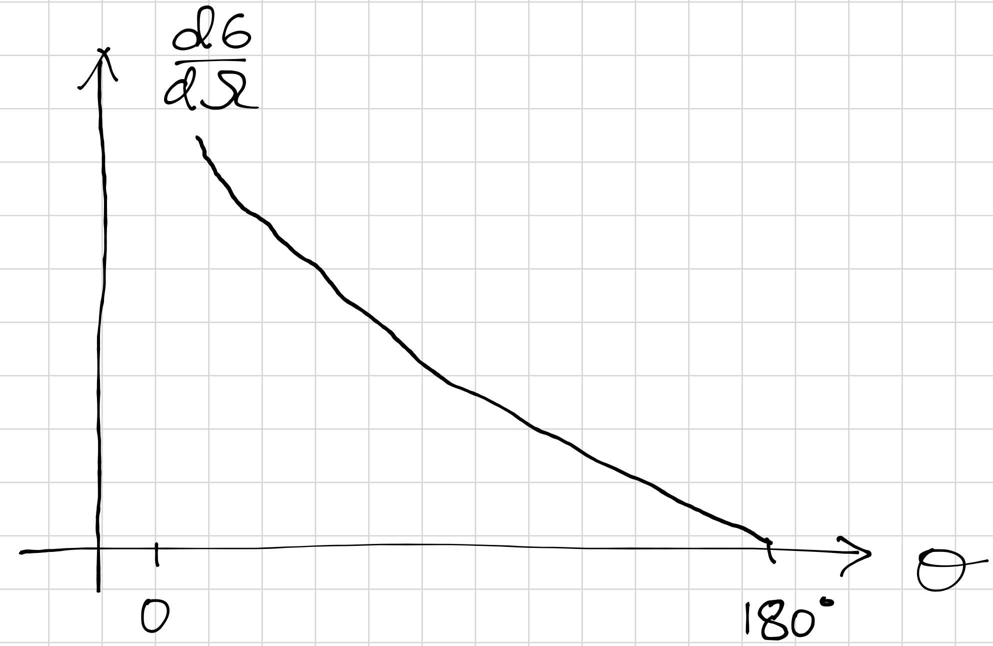
$$\frac{d\sigma}{d\Omega_{cm}} = \frac{1}{64\pi^2 s}\,|\overline{\mathcal{M}}|^2 = \frac{Z^2\alpha^2}{4E_e^2}\,\frac{\cos^2\theta/2}{\sin^4\theta/2}$$

$\hookrightarrow$ Rutherford or Mott formula

Difference : Rutherford in Lab frame

$$E' = E$$

$$\frac{E'}{E^3}$$



How this pattern changes ~~for~~ spin-½ ?

$$|\overline{\mathcal{M}}|^2 = \frac{Z^2 e^2}{t^2}\, \ell^{\mu\nu} h^b_{\mu\nu}$$

$$h^b_{\mu\nu} = 2\left[p'_\mu p_\nu + p'_\nu p_\mu + \tfrac{t}{2} g_{\mu\nu} \right]$$

$$\underline{h^b_{\mu\nu} - h^a_{\mu\nu}} = \cancel{2} p'_\mu p_\nu + \cancel{2} p'_\nu p_\mu + t g_{\mu\nu}$$

$$- p_\mu p_\nu - \cancel{p'_\mu p_\nu} - p'_\mu p'_\nu - \cancel{p_\mu p'_\nu}$$

$\Delta^\mu = p'^\mu - p^\mu$

$\Delta^2 = t$

$$= \Delta_\mu p_\nu - p'_\nu \Delta_\mu + t g_{\mu\nu}$$

$$= -\Delta_\mu \Delta_\nu + \Delta^2 g_{\mu\nu}$$

$$\ell^{\mu\nu} = \underbrace{(k+k')^\mu (k+k')^\nu}_{\text{spin-0}} \underbrace{- \Delta^\mu \Delta^\nu + \Delta^2 g^{\mu\nu}}_{\text{spin-}1/2\text{ correction}}$$

$$\Delta_\mu \ell^{\mu\nu} = \Delta_\nu \ell^{\mu\nu} = 0$$

$$\ell^{\mu\nu} \cdot \left(h^b_{\mu\nu} - h^a_{\mu\nu} \right) = \Delta^2 \ell^{\mu\nu} g_{\mu\nu} = \Delta^2 (4t + 4kk')$$

$$= 2t^2$$

$t = -4E_e^2 \sin^2\frac{\theta}{2}$

$$\ell^{\mu\nu} \cdot h^b_{\mu\nu} = 16\, s\, E_e^2 \cos^2\frac{\theta}{2} + 32 E_e^4 \sin^4\frac{\theta}{2}$$

$$\frac{d\sigma^{1/2}}{d\Omega_{cm}} = \frac{Z^2\alpha^2}{4E_e^2}\frac{1}{\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2} - \frac{t}{2s}\sin^2\frac{\theta}{2}\right]$$

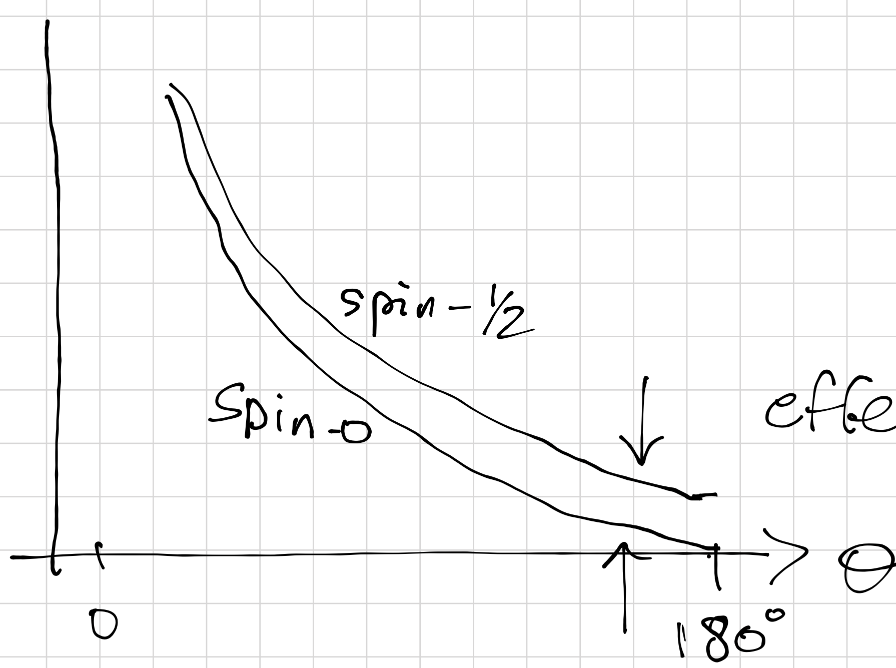
$$s = M^2 + 2\sqrt{s}\,E_e$$

$$s - 2\sqrt{s}\,E_e - M^2 = 0$$

$$\sqrt{s} = E_e + \sqrt{E_e^2 + M^2} = M + E_e + \cancel{\frac{E_e^2}{2M} + \dots}$$

$E \ll M$ $\qquad\qquad \simeq M$

The difference : recoil $\sim \frac{t}{2M^2}$



What is the origin?

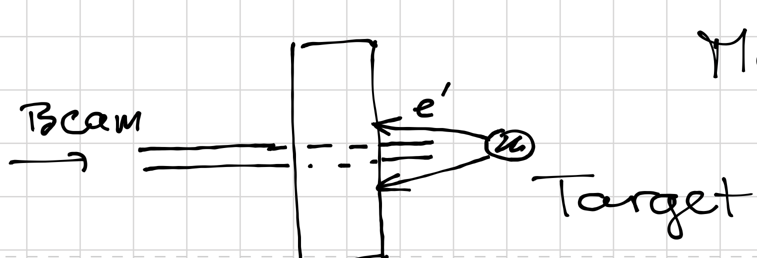
$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p+p')^\mu}{2M} + \frac{i\sigma^{\mu\alpha}(p'-p)_\alpha}{2M}\right]u(p)$$

Gordon ID

spin-0 $\qquad$ magnetic mom. of a Dirac particle

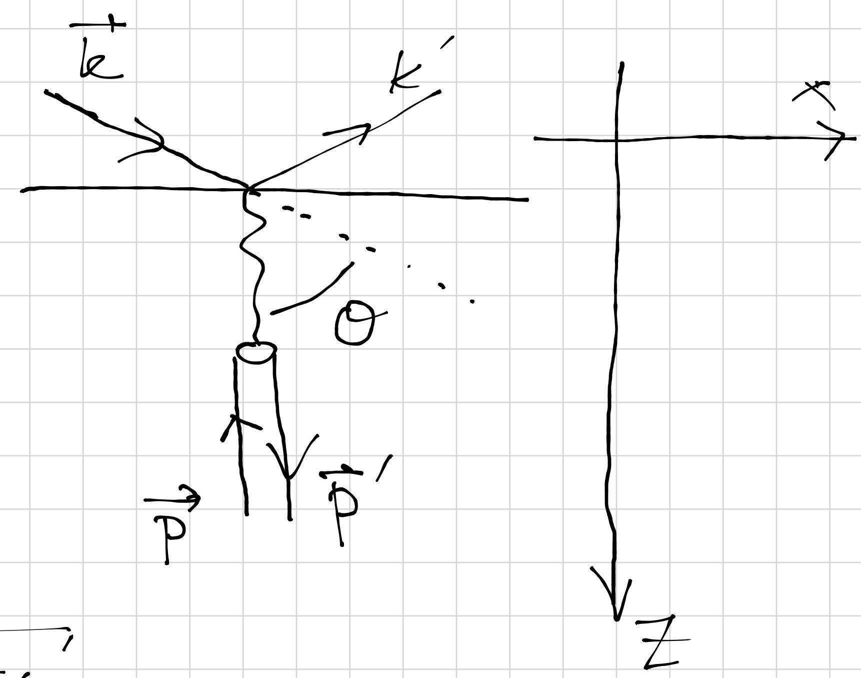
$$\mu_D = \frac{e}{2M}$$

Magnetic scattering



"Helicity" formalism in the Breit frame

Breit (brick-wall)



Def. $\vec{p} + \vec{p}\,' = 0$

$$p^\mu = \left(P^0, 0, 0, -\frac{\Delta}{2}\right)$$
$$p'^\mu = \left(P^0, 0, 0, +\frac{\Delta}{2}\right)$$
$$(p'-p)^\mu = \Delta^\mu \qquad P^0 = \sqrt{M^2 - t/4}$$

$$k^\mu = E_B(1, \sin\theta, 0, \cos\theta)$$
$$k'^\mu = E_B(1, \sin\theta, 0, -\cos\theta)$$

$$\underline{2E_B \cos\theta = \Delta}$$

$$(p+p'+k+k')^2 = 4s$$

Nucleon spinors : quantize spin in z-dir.

$\lambda = 2\underline{s_z}$

$$u_\lambda(p) = \sqrt{P^0+M}\begin{pmatrix} \varepsilon_\lambda \\ \frac{\vec{\sigma}\vec{p}}{P^0+M}\varepsilon_\lambda \end{pmatrix}$$

ε_λ: $\varepsilon_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$ $\varepsilon_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$

$$= \sqrt{P^0+M}\begin{pmatrix} \varepsilon_\lambda \\ -\lambda\frac{\Delta/2}{P^0+M}\varepsilon_\lambda \end{pmatrix}$$

$$u_{\lambda'}(p') = \sqrt{P^0+M}\begin{pmatrix} \varepsilon_{\lambda'} \\ +\lambda'\frac{\Delta/2}{P^0+M}\varepsilon_{\lambda'} \end{pmatrix}$$

$$\bar{u}(k')\gamma^\mu u(k)\cdot\bar{u}(p')\left[\frac{(p+p')_\mu}{2M} + \frac{i\sigma_{\mu\alpha}\Delta^\alpha}{2M}\right]u(p)$$

spin-0 comp. $\frac{2P^0}{2M}\bar{u}(p')u(p)\cdot\underline{\bar{u}(k')\gamma^0 u(k)}$

$$\bar{u}_{\lambda'}(p')\, u_\lambda(p) = (P^0+M)\Big(1 + \underbrace{\lambda\lambda'}_{\substack{\parallel \\ +1}} \frac{\Delta^2/4}{(P^0+M)^2}\Big)\, \underbrace{\xi^+_{\lambda'}\xi_\lambda}_{\delta_{\lambda\lambda'}}$$

$$\frac{\Delta^2/4}{(P^0+M)^2} \to \frac{P^{0\,2}-M^2}{(P^0+M)^2}$$

$$= 2P^0\,\delta_{\lambda\lambda'}$$

Electrons : helicity repr.

$$u_h(k) = \sqrt{E_B}\begin{pmatrix} \xi_h(\hat{k}) \\ \frac{\vec{\sigma}\vec{k}}{E_B}\cdot \xi_h(\hat{k}) \end{pmatrix} = \sqrt{E_B}\begin{pmatrix} \xi_h(\hat{k}) \\ 2h\,\xi_h(\hat{k}) \end{pmatrix}$$

$$u_{h'}(k') = \sqrt{E_B}\begin{pmatrix} \xi_{h'}(\hat{k}') \\ 2h'(\hat{k}') \end{pmatrix}$$

$\hookrightarrow$ spin-0 part $\longrightarrow$ $2P^0\,\delta_{\lambda\lambda'}\cdot \frac{P^0}{M}\, 2E_B\,\delta_{hh'}\,\xi^+_{h'}\xi_h$

$$u^+(k')\,u(k) = E_B\left(\xi^+\xi + 2h\cdot 2h'\,\xi^+\xi\right) = E_B\,\underline{(1+4hh')}\,\xi^+\xi$$

$$\xi_h(\hat{h}) = \begin{cases} h=+1/2 & \xi_\uparrow(\theta) = \begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix} \\ h=-1/2 & \xi_\downarrow(\theta) = \begin{pmatrix}-\sin\theta/2 \\ \cos\theta/2\end{pmatrix}\end{cases}$$

$$\xi_{h'}(\hat{h}') = \begin{cases} h=+1/2 & \xi_\uparrow(\pi-\theta) = \begin{pmatrix}\sin\theta/2 \\ \cos\theta/2\end{pmatrix} \\ h=-1/2 & \xi_\downarrow(\pi-\theta) = \begin{pmatrix}-\cos\theta/2 \\ \sin\theta/2\end{pmatrix}\end{cases}$$

$(\sin\theta, 0, -\cos\theta)$

$\theta \to \pi-\theta$

$$i\sigma^{\mu\alpha}\underset{\substack{\Delta_3 \\ \parallel}}{\Delta_\alpha} = -\Delta^3\, i\sigma^{\mu 3} = -\Delta^3\left(-\frac{1}{2}\right)\left[\gamma^\mu\gamma^3 - \gamma^3\gamma^\mu\right]$$

$\mu = 0,1,2$

$$= \frac{\Delta}{2}\left(\gamma^\mu\gamma^3 - \gamma^3\gamma^\mu\right)$$

$$\bar{u}(-i)\sigma^{03}u = \underline{u^{+}\gamma^{3}u} = u^{+}_{\lambda'}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} u_\lambda$$

$$= (P^0+M)\left(-\lambda\frac{\Delta/2}{P^0+M} - \lambda'\frac{\Delta/2}{P^0+M}\right)$$

$$\cdot\, \xi^{+}_{\lambda'}\underbrace{\sigma^3\xi_\lambda}_{\to -\lambda\xi_\lambda}$$

$$= +\Delta\,\underline{\delta_{\lambda\lambda'}}$$

$$(1,2) \longrightarrow \quad \sigma^{\pm} = \frac{1}{\sqrt{2}}(\sigma^1 \pm i\sigma^2)$$

$$\frac{1}{2}(\gamma^{\pm}\gamma^{3} - \gamma^{3}\gamma^{\pm}) = -\begin{pmatrix} \sigma^{\pm}\sigma^{3} - \sigma^{3}\sigma^{\pm} & 0 \\ 0 & -\,\text{''}\,- \end{pmatrix}$$

$$\begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$$

$$\underline{\sigma^3\sigma^{\pm} = \pm\sigma^{\pm}}$$

$$\sigma^{+} = \frac{1}{\sqrt{2}}(\sigma^1 + i\sigma^2) = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$$

$$\sigma^{-} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}$$

$$u^{+}_{\lambda'}\gamma^{0}\begin{pmatrix} \sigma^{\pm} & 0 \\ 0 & \sigma^{\pm} \end{pmatrix} u_\lambda = (P^0+M)\left[1 + \lambda\lambda'\frac{\Delta^2/4}{(P^0+M)^2}\right]$$

$$\cdot\, \underbrace{\xi^{+}_{\lambda'}\sigma^{\pm}\xi_\lambda}$$

$$\downarrow$$

$$\sim \underline{\delta_{\lambda'-\lambda}}$$

$$\sigma^{+}\cdot\xi_{+1} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\rightarrow \quad \sigma^{-}\cdot\xi_{+1} = \xi_{-1}\cdot\sqrt{2}$$

$$\rightarrow \quad \sigma^{+}\cdot\xi_{-1} = \xi_{+1}$$

$$\sigma^{-}\cdot\xi_{-1} = 0$$

$$\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} \equiv \vec{\gamma}_{\perp}$$

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= a^+ b^- + a^- b^+ + a_3 b_3$$

$$\bar{u}(k')\gamma^\mu u(k) \cdot \bar{u}(p')\, i\sigma^{\mu 3} u(p)$$

$$= \bar{u}(k')\gamma^0 u(k)\, \bar{u}(p')\, i\sigma^{03} u(p)$$

$$- \bar{u}(k')\gamma^+ u(k)\, \bar{u}(p')\, i\sigma^{-3} u(p)$$

$$- \bar{u}(k')\gamma^- u(k)\, \bar{u}(p')\, i\sigma^{+3} u(p)$$

Ex

$\mathcal{M}^{h'h}_{\lambda'\lambda}$

1. Electron hel. conserved

$h' = h = +1/2 \qquad h' = h = -1/2$

$\mathcal{M}^{++}_{++}$; $\mathcal{M}^{++}_{--}$; $\mathcal{M}^{--}_{++}$; $\mathcal{M}^{--}_{--}$

$\mathcal{M}^{++}_{+-}$; $\mathcal{M}^{++}_{-+}$; $\mathcal{M}^{--}_{+-}$; $\mathcal{M}^{--}_{-+}$