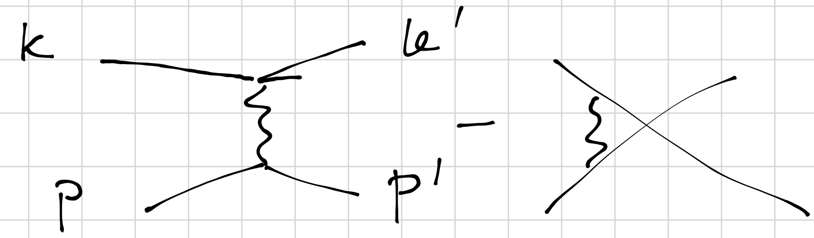
Lecture 21



$$\mathcal{M}_{\text{Møller}} = \frac{e^2}{t}\,\bar{u}(k')\gamma^\mu u(k)\;\bar{u}(p')\gamma_\mu u(p) \longrightarrow \mathcal{M}_t$$

$$- \frac{e^2}{u}\,\bar{u}(k')\gamma^\mu u(p)\;\bar{u}(p')\gamma_\mu u(k) \longrightarrow \mathcal{M}_u$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,\frac{1}{4}|\mathcal{M}|^2$$

$$\left(\bar{u}(k')\gamma^\mu u(k)\right)^* = \bar{u}(k)\gamma^\mu u(k')$$

$$|\mathcal{M}|^2 = \mathcal{M}^*\mathcal{M}$$

$$\frac{1}{4}|\mathcal{M}_t|^2 = \frac{e^4}{t^2}\,\frac{1}{4}\,\mathrm{Tr}\left[(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma^\nu\right]$$

$$\mathrm{Tr}\left[(\not{p}'+m)\gamma_\mu(\not{p}+m)\gamma_\nu\right]$$

$$= \frac{e^4}{t^2}\,K^{\mu\nu}\cdot P_{\mu\nu}$$

$$K^{\mu\nu} = 2\left(k^\mu k'^\nu + k^\nu k'^\mu + \frac{t}{2}g^{\mu\nu}\right)$$

$$P^{\mu\nu} = 2\left(p^\mu p'^\nu + p^\nu p'^\mu + \frac{t}{2}g^{\mu\nu}\right)$$

$$\hookrightarrow \frac{1}{4}|\mathcal{M}_t|^2 = \frac{4e^4}{t^2}\Big[2(pk)(p'k') + 2(pk')(p'k)$$

$$t^2 + t\,\underbrace{(kk'+pp')}_{=2m^2-t}\Big]$$

$$s = (p+k)^2 = (p'+k')^2 = 2m^2 + 2pk = 2m^2 + 2p'k'$$
$$u = (p-k')^2 = (p'-k)^2 = 2m^2 - 2pk' = 2m^2 - 2p'k$$
$$t = (k-k')^2 = (p'-p)^2 = 2m^2 - 2kk' = 2m^2 - 2pp'$$

$$\hookrightarrow \quad \frac{1}{4}|\mathcal{M}_t|^2 = \frac{2e^4}{t^2}\left[(s-2m^2)^2 + (2m^2-u)^2 + 4m^2 t\right]$$
$$= \frac{2e^4}{t^2}\left[s^2 + u^2 - 4m^2(s+u-t) + 8m^4\right]$$

$$\frac{1}{4}|\mathcal{M}_u|^2 = \frac{2e^4}{u^2}\left[s^2 + t^2 - 4m^2\underbrace{(s-u+t)} + 8m^4\right]$$

$s+t+u = 4m^2$

$s+t-u = -2u + 4m^2$

$$\frac{1}{4}\left(\mathcal{M}_t^* \mathcal{M}_u + \mathcal{M}_u^* \mathcal{M}_t\right)$$
$$= -\frac{e^4}{2tu}\sum_{\text{spins}} \left(\bar{u}(p)\gamma^\mu u(k')\right)\left(\bar{u}(k)\gamma_\mu u(p')\right)$$
$$\cdot \left(\bar{u}(k')\gamma^\nu u(k)\right)\left(\bar{u}(p')\gamma_\nu u(p)\right)$$

$$\sum u_p \bar{u}_p \rightarrow (\not{p}+m)$$

$$= -\frac{e^4}{2tu}\mathrm{Tr}\left[(\not{p}+m)\gamma^\mu(\not{k}'+m)\gamma^\nu(\not{k}+m)\gamma_\mu(\not{p}'+m)\gamma_\nu\right]$$

$$*)\ \gamma^\mu\left(\not{k}'\gamma^\nu\not{k} + m^2\gamma^\nu + m(\not{k}'\gamma^\nu + \gamma^\nu\not{k})\right)\gamma_\mu$$
$$= -2\not{k}\gamma^\nu\not{k}' - 2m^2\gamma^\nu + 4m(k+k')^\nu$$

$$= -\frac{e^4}{2tu}\mathrm{Tr}\ (\not{p}+m)\left[4m(k+k')^\nu - 2m^2\gamma^\nu - 2\not{k}\gamma^\nu\not{k}'\right](\not{p}'+m)\gamma_\nu$$

$$= -\frac{e^4}{2tu}\Big\{ 4m\,\mathrm{Tr}(\not{p}+m)(\not{p}'+m)(\not{k}+\not{k}') \longrightarrow 4m(p+p', k+k')$$

$$-2m^2\,\mathrm{Tr}(\not{p}+m)\underbrace{\gamma^\nu(\not{p}'+m)\gamma_\nu}_{=4m-2\not{p}'} \longrightarrow 16m^2 - 8pp'$$

$$-2\,\mathrm{Tr}(\not{p}+m)\not{k}\underbrace{\gamma^\nu\not{k}'(\not{p}'+m)\gamma_\nu}_{=4p'k'-2m\not{k}'} \longrightarrow 4(p'k')4(pk) - 8m^2(pk')\Big\}$$

$$= -\frac{e^4}{2tu}\Big[16m^2\underbrace{(pk+p'k'+pk'+p'k)}_{\to s-u}$$

$$-\cancel{32m^4} + 16m^2\left(\cancel{m^2}-\frac{t}{2}\right) - 4(s-2m^2)^2$$

$$+16m^2\left(\cancel{m^2}-\frac{u}{2}\right)\Big]$$

$$= -\frac{e^4}{2tu}\left[-4(s-2m^2)^2 + 16m^2(s-u) - 8m^2(t+u)\right]$$

$$= \frac{2e^4}{tu}\left[s^2 - 4m^2 s + 4m^4 + 2m^2(t+u) - 4m^2(s-u)\right]$$

$$\vdots$$

$$= \frac{4e^4}{tu}\left[s^2 - 8m^2 s + 12m^4\right]$$

$$\Downarrow$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{2e^4}{t^2}\left[s^2+u^2+8m^2t-8m^4\right]$$

$$+\frac{2e^4}{u^2}\left[s^2+t^2+8m^2u-8m^4\right]$$

$$+\frac{4e^4}{tu}\left[s^2-8sm^2+12m^4\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,\frac{1}{4}\sum |\mathcal{M}|^2$$

c.m. frame

$p = (E, 0, 0, p)$ $p = \sqrt{\frac{s}{4} - m^2}$

$k = (E, 0, 0, -p)$ $E = \frac{\sqrt{s}}{2}$

$p' = (E, p\sin\theta, 0, p\cos\theta)$

$k' = (E, -\vec{p}\,')$

$s = 4E^2$

$t = -2p^2(1-\cos\theta) = -4p^2\sin^2\frac{\theta}{2}$

$u = -2p^2(1+\cos\theta) = -4p^2\cos^2\frac{\theta}{2}$

$e^2 = 4\pi\alpha$, $\alpha \simeq \frac{1}{137}$ e.m. const. $\frac{2e^4}{64\pi^2} = \frac{\alpha^2}{2}$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\,\frac{16E^4 + 16p^4\cos^4\frac{\theta}{2} - 32m^2p^2\sin^2\frac{\theta}{2} - 8m^4}{4p^4(1-\cos\theta)^2}$$

$$+ \frac{\alpha^2}{2s}\,\frac{16E^4 + 16p^4\sin^4\frac{\theta}{2} - 32p^2m^2\cos^2\frac{\theta}{2} - 8m^4}{4p^4(1+\cos\theta)^2}$$

$$+ 2\,\frac{\alpha^2}{2s}\,\frac{16E^4 - 32m^2E^2 + 12m^4}{4p^4\sin^2\theta}$$

$$= \frac{2\alpha^2}{s}\,\frac{1}{p^4\sin^4\theta}\Big[\Big(E^4 + p^4\cos^4\frac{\theta}{2} - 8m^2p^2\sin^2\frac{\theta}{2} - \frac{m^4}{2}\Big)(1+\cos\theta)^2$$

$$+\Big(E^4 + p^4\sin^4\frac{\theta}{2} - 8m^2p^2\cos^2\frac{\theta}{2} - \frac{m^4}{2}\Big)(1-\cos\theta)^2$$

$$+2\Big(E^4 - 2m^2E^2 + \frac{3}{4}m^4\Big)\sin^2\theta\Big]$$

Study 1. Non-relativistic case

$p \ll m$, $E \simeq m$

$$\frac{d\sigma}{d\Omega} \longrightarrow \frac{2\alpha^2}{4m^2}\frac{1}{p^4 \sin^4\theta}\Big[\frac{m^4}{2}(1+\cos\theta)^2 + \frac{m^4}{2}(1-\cos\theta)^2$$

$$+ \underbrace{2\left(m^4 - 2m^4 + \tfrac{3}{4}m^4\right)}_{= -\frac{m^4}{2}} \sin^2\theta$$

$$= \frac{\alpha^2}{4p^4\sin^4\theta}\Big[\underbrace{(1+\cos\theta)^2 + (1-\cos\theta)^2 - \sin^2\theta}_{= 1+3\cos^2\theta}\Big]$$

Scalar case:

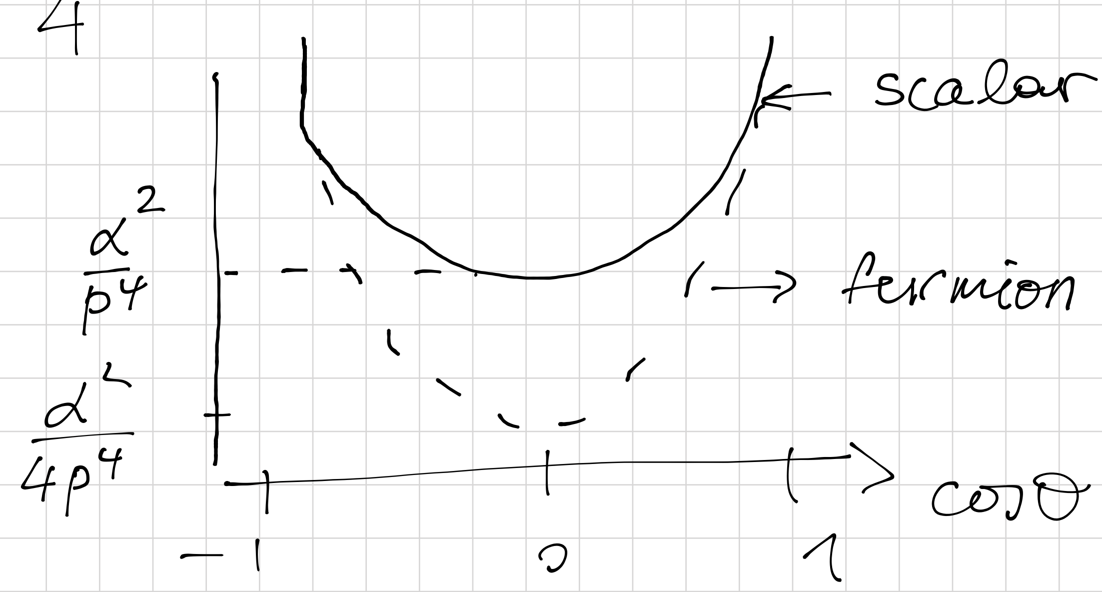
$$\mathcal{M} = \frac{e^2}{t}(s-u) + \frac{e^2}{u}(s-t)$$

$$= -\frac{e^2 \cdot 4E^2}{2p^2(1-\cos\theta)} - \frac{e^2\cdot 4E^2}{2p^2(1+\cos\theta)}$$

$$= -4E^2e^2\frac{1}{p^2\sin^2\theta}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\cdot 16E^4(4\pi\alpha)^2\frac{1}{p^4\sin^4\theta} = \frac{\alpha^2}{p^4\sin^4\theta}$$

$$\frac{1+3\cos^2\theta}{4} \longleftrightarrow 1$$



2. Ultra relativistic case

$$m \ll p \qquad p \simeq E = \frac{\sqrt{s}}{2}$$

$$\frac{2\alpha^2}{s}\frac{\cancel{E^4}}{\cancel{p^4}\sin^4\theta}\Big[\left(1+\cos^4\tfrac{\theta}{2}\right)(1+\cos\theta)^2 + \left(1+\sin^4\tfrac{\theta}{2}\right)(1-\cos\theta)^2 + 2\sin^2\theta\Big]$$

$$= \frac{\alpha^2}{s\sin^4\theta}\left(3+\cos^2\theta\right)^2$$

Scalar:

$$s-u = s+\frac{s}{2}(1+\cos\theta) = \frac{s(3+\cos\theta)}{2}$$

$$s$$

$$\mathcal{M} = -e^2\left[\frac{3+\cos\theta}{1-\cos\theta} + \frac{3-\cos\theta}{1+\cos\theta}\right]$$

$$= -2e^2\,\frac{3+\cos^2\theta}{\sin^2\theta}$$

$$\hookrightarrow \frac{\alpha^2}{s\sin^4\theta}\left(3+\cos^2\theta\right)^2$$

Ultrarel. case $\longrightarrow$ helicity is conserved

Weyl representation $\quad u_\uparrow = \sqrt{2E}\begin{pmatrix} 0 \\ \xi_\uparrow \end{pmatrix}$

$$\underline{\gamma^0\gamma^\mu} = \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \qquad u_\downarrow = \sqrt{2E}\begin{pmatrix} \xi_\downarrow \\ 0 \end{pmatrix}$$

$$\xi_\uparrow(\vec{k}(\theta,\varphi)) = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix}$$

$$\sigma^\mu = (1,\vec{\sigma}) \quad \bar{\sigma}^\mu = (1,-\vec{\sigma}) \qquad \xi_\downarrow = \begin{pmatrix} -e^{-i\varphi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$$\bar{u}_\uparrow(k')\gamma^\mu u_\uparrow(k) = 2E\, \xi_\uparrow^\dagger(k')\,\sigma^\mu\, \xi_\uparrow(k)$$
$$\bar{u}_\downarrow(k')\gamma^\mu u_\downarrow(k) = 2E\, \xi_\downarrow^\dagger(k')\,\bar{\sigma}^\mu\, \xi_\downarrow(k)$$

Helicity formalism:

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \sum_{\text{helicities}} \left|\mathcal{M}(h_k, h_p; h_{k'} h_{p'})\right|^2$$

$$\mathcal{M}_t = \frac{e^2}{t}\, \bar{u}(k')\gamma^\mu u(k)\; \bar{u}(p')\gamma_\mu u(p)$$

How many hel. ampl. are there?

URel. $\rightarrow$ helicity conserved

$$\underset{k\,p\;\,k'\,p'}{\mathcal{M}_{++;++}} \qquad \mathcal{M}_{--;--} \qquad \mathcal{M}_{+-,+-} \qquad \mathcal{M}_{-+,-+}$$

$$\xi(\vec{p}(0,0)) \rightarrow \quad \xi_\uparrow^p = \begin{pmatrix}1\\0\end{pmatrix} \quad \xi_\downarrow^p = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\xi(\vec{p}'(\theta,0)) \rightarrow \quad \xi_\uparrow^{p'} = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} \quad \xi_\downarrow^{p'} = \begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix}$$

$$\xi(\vec{k}(\pi,0)) \rightarrow \quad \xi_\uparrow^k = \begin{pmatrix}0\\1\end{pmatrix} \quad \xi_\downarrow^k = \begin{pmatrix}-1\\0\end{pmatrix}$$

$$\xi(k'(\pi+\theta,0)) \rightarrow \quad \xi_\uparrow^{k'} = \begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix} \quad \xi_\downarrow^{k'} = \begin{pmatrix}-\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\end{pmatrix}$$

$$\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = -\sin\frac{\theta}{2}$$
$$\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = +\cos\frac{\theta}{2}$$

$$\bar{u}_\uparrow(k')\gamma^\mu u_\uparrow(k) = 2E\ \xi_\uparrow^{k'+}\sigma^\mu\xi_\uparrow^{k}$$

Ex.

$$= 2E\begin{pmatrix}\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\\ i\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}$$

$$\bar{u}_\downarrow(k')\gamma^\mu u_\downarrow(k) = 2E\begin{pmatrix}\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}$$

$$\bar{u}_\uparrow(p')\gamma^\mu u_\uparrow(p) = 2E\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\\ i\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$$

$$\bar{u}_\downarrow(p')\gamma^\mu u_\downarrow(p) = 2E\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$$

$$\boxed{\begin{aligned}\mathcal{M}^t_{++,++} &= \frac{e^2}{t}\cdot 8E^2 = \mathcal{M}^t_{--,--}\\ \mathcal{M}^t_{+-,+-} &= \mathcal{M}^t_{-+,-+} = \frac{e^2}{t}8E^2\cos^2\frac{\theta}{2}\end{aligned}}$$

$$\mathcal{M}^u = -\frac{e^2}{u}\bar{u}(k')\gamma^\mu u(p)\ \ \bar{u}(p')\gamma_\mu u(k)$$

$$u(k') = u(p'(\theta\to\pi+\theta)) :$$

$$\bar{u}(k')\gamma^\mu u(p) = \bar{u}(p')\gamma^\mu u(p) : \left| \begin{array}{l} \cos\frac{\theta}{2} \to -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} \to \cos\frac{\theta}{2} \end{array} \right.$$

$$u(p') = -u(k'(\pi+\theta \to 2\pi+\theta))$$

$$\bar{u}(p')\gamma^\mu u(k) = \bar{u}(k')\gamma^\mu u(k) : \left| \begin{array}{l} \cos\frac{\theta}{2} \to +\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} \to -\cos\frac{\theta}{2} \end{array} \right.$$

$\downarrow$

$$\mathcal{M}^u_{++,++} = \mathcal{M}^u_{--,--} = \frac{e^2}{u} 8E^2$$

$$\mathcal{M}^u_{+-,-+} = \mathcal{M}^u_{-+,+-} = \frac{e^2}{u} 8E^2 \sin^2\frac{\theta}{2}$$

$\mathcal{M}^{u+t}$:

$$\frac{1}{t} + \frac{1}{u} = \frac{-1}{E^2\sin^2\theta}$$

$$\mathcal{M}_{++,++} = \mathcal{M}_{--,--} = \frac{-8E^2e^2}{E^2\sin^2\theta}$$

$$\mathcal{M}^t_{+-,+-} = \mathcal{M}^t_{-+,-+} = -2E^2\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}$$

$$\mathcal{M}^u_{-+,+-} = \mathcal{M}^u_{+-,-+} = -2E^2\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$$

$\downarrow$

$$\frac{1}{4}\sum_{h_k,h_{k'},h_p,h_{p'}} \left|\mathcal{M}_{h_k h_p; h_{k'} h_{p'}}\right|^2 = \frac{64\pi^2\alpha^2}{\sin^4\theta} \times (3+\cos^2\theta)^2$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s\sin^4\theta}(3+\cos^2\theta)^2$$

Non-relativistic case

Weyl spinors $\longrightarrow$ $u \simeq \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$

Dirac spinors $u = \sqrt{E+m}\begin{pmatrix} \xi \\ \frac{\vec{\sigma}\vec{p}}{E+m}\xi \end{pmatrix} \rightarrow \sqrt{2m}\begin{pmatrix} \xi \\ 0 \end{pmatrix}$

$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ $\quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$

$\bar{u}\gamma^\mu u = u^+ \gamma^0 \gamma^\mu u = u^+ \begin{pmatrix} \mathbb{1} \\ \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \end{pmatrix} u$

NR limit $\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ mixes upper and lower comp.

NR: $\bar{u}\gamma^\mu u = \begin{pmatrix} u^+ u \\ \vec{0} \end{pmatrix}$