Lecture 20

Last lecture: interacting Fermion field

→ scalar
pseudoscalar } int.

↳ obtained tree-level ampl.
for $e^-e^- \to e^-e^-$
$e^+e^- \to e^+e^-$

Devised the respective Feynman rules

Spinor QED

$$\boxed{\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA^{\mu}\bar{\psi}\gamma_{\mu}\psi}$$

Conserved current $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$

$$\mathcal{L}_{Free} = \bar{\psi}(i\not{\partial} - m)\underline{\psi}$$

$$\frac{\psi \to e^{i\alpha}\psi}{\mathcal{L} \to \mathcal{L}}$$

$$= \partial_{\mu} i \underbrace{\bar{\psi}\gamma^{\mu}\psi}$$

$$\mathcal{L}_{int} = A^{\mu} j_{\mu}$$

$\phi^{+}\overleftrightarrow{\partial}^{\mu}\phi$ (scalar)

Gauge invariance

$$\hookrightarrow A^{\mu} \longrightarrow A^{\mu} + \partial^{\mu}\lambda(x)$$

$$\begin{cases} \psi \longrightarrow e^{-ie\lambda(x)}\,\psi \\ \bar\psi \longrightarrow \bar\psi\, e^{+ie\lambda(x)} \end{cases}$$

$$D_\mu \psi = (\partial_\mu + ieA_\mu)\psi$$

$$\bar\psi\, e^{ie\lambda(x)}(i\partial\!\!\!/ - m)\, e^{-ie\lambda(x)}\psi$$

$$= \bar\psi(i\partial\!\!\!/ - m)\psi + \cancel{\underbrace{(\partial_\mu\lambda)\, e\,\bar\psi\gamma^\mu\psi}}$$

$$- e(A^\mu + \cancel{\underbrace{\partial^\mu\lambda}})\,\bar\psi\gamma_\mu\psi$$

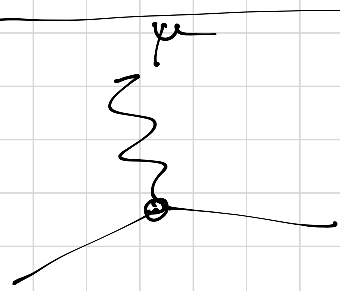
$$D\!\!\!/ = D_\mu\gamma^\mu$$

$$\underline{D_\mu\psi = (\partial_\mu + ieA_\mu)}\,\psi$$

$$\longmapsto e^{-ie\lambda(x)} D_\mu\psi$$

$$\mathcal{L} = \bar\psi(iD\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Spinor QED Feynman rule



$\longrightarrow$ $\underline{-ie\gamma^\mu}$

if $\vec\partial\cdot\vec j = 0$

$\partial_t j^0 = 0$

$\partial_\mu j^\mu = 0$

Conserved current $j^\mu = e\bar\psi\gamma^\mu\psi$

—//— charge $J^0 = e\int d^3x\, \bar\psi\gamma^0\psi = Q$

$$= e\int \frac{d^3\tilde p}{(2\pi)^3}\sum_s\left[b_p^{s+}b_p^s - c_p^{s+}c_p^s\right]$$

part. − # anti-part.

Charge conjugation C

$$(i\not\partial - m - e\not A)\psi = 0 \qquad \not\partial = \partial_\mu \gamma^\mu$$

$$\psi^{(c)} = C\psi^*$$

Complex conj. of D.Eq.

$$(-i\partial_\mu \gamma^{\mu *} - m - e\gamma^{\mu *}A_\mu)\psi^* = 0$$

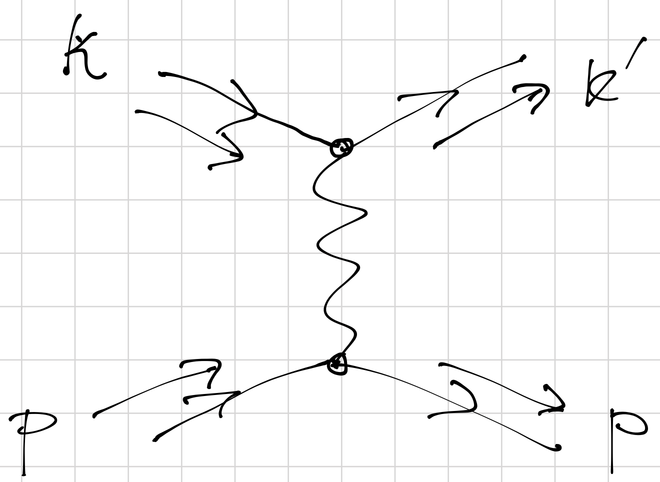
$$C^\dagger \gamma^\mu C = -\gamma^{\mu *} \qquad C^\dagger C = \mathbb{1}$$

$$C^\dagger(i\not\partial C - mC + e\not A C)\psi^*$$

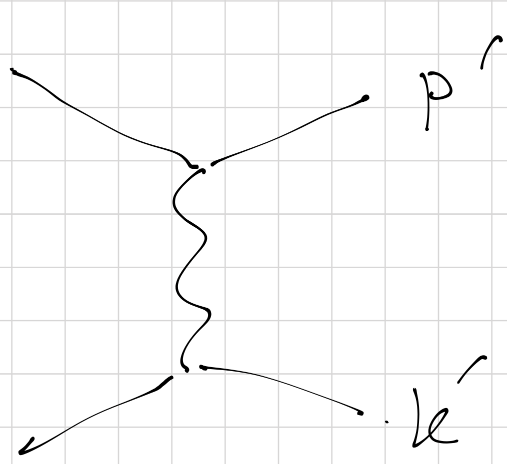
$$\hookrightarrow (i\not\partial - m + e\not A)\psi^{(c)} = 0$$

Let's consider Møller scattering

$$e^-e^- \longrightarrow e^-e^-$$



$$= (-ie)^2\, \bar u(k')\gamma^\mu u(k)\, \frac{-ig_{\mu\nu}}{(k-k')^2+i\varepsilon} \cdot \bar u(p')\gamma^\nu u(p)$$



add. "−" due to Fermi stat.

$$-(-ie)^2\, \bar u(p')\gamma^\mu u(k)\, \frac{-ig_{\mu\nu}}{(k-p')^2+i\varepsilon}\, \bar u(k')\gamma^\nu u(p)$$

$$\boxed{A_{\text{Moller}} = e^2\left[\frac{\bar{u}(k')\gamma^\mu u(k)\ \bar{u}(p')\gamma_\mu u(p)}{(k-k')^2+i\varepsilon} - \frac{\bar{u}(p')\gamma^\mu u(k)\ \bar{u}(k')\gamma_\mu u(p)}{(k-p')^2+i\varepsilon}\right]}$$

Physical region of scattering:

$$s \geq (2m)^2$$

$$s = (p+k)^2 = (p'+k')^2$$
$$u = (p-k')^2 = (k-p')^2$$
$$t = (k-k')^2 = (p-p')^2$$

c.m. frame

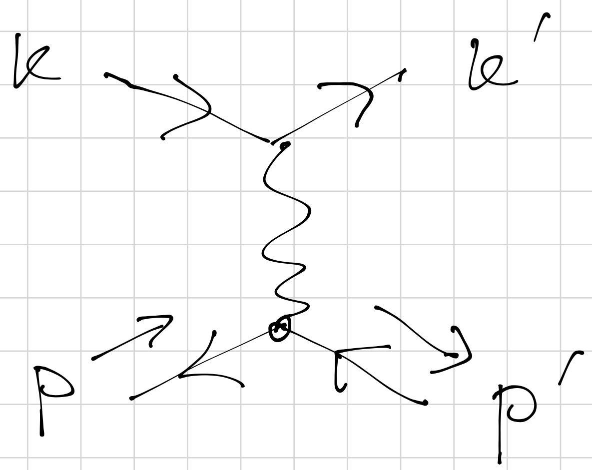
$$\vec{p}+\vec{k} = \vec{p}\,'+\vec{k}\,' = 0$$
$$E_p = E_k = E_{p'} = E_{k'} = \frac{\sqrt{s}}{2}$$

$$(k-k')^2 = -(\vec{k}-\vec{k}\,')^2 < 0$$
$$(p-k')^2 = -(\vec{p}-\vec{k}\,')^2 < 0$$

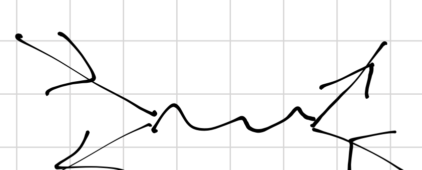
Recall scalar QED: $ie^2\dfrac{(k+k')_\mu (p+p')^\mu}{(k-k')^2+i\varepsilon}$

$$+ie^2\frac{(k+p')_\mu (p+k')^\mu}{(k-p')^2+i\varepsilon}$$

Bhabha scattering $\quad e^+e^- \to e^+e^-$



$$-(-ie)^2\,\bar{u}(k')\gamma^\mu u(k)\,\frac{-i g_{\mu\nu}}{(k-k')^2+i\varepsilon}\,\bar{v}(p)\gamma^\nu v(p')$$



$$\frac{e^2}{(p+k)^2+i\varepsilon}\,\bar{v}(p)\gamma^\mu u(k)\ \bar{u}(k')\gamma_\mu v(p')$$

We want to compute cross section for Moller scattering

$$A_{\text{Møller}} = \frac{e^2}{t}\,\bar{u}(k')\gamma^\mu u(k)\,\bar{u}(p')\gamma_\mu u(p)$$

$$-\frac{e^2}{u}\,\bar{u}(k')\gamma^\mu u(p)\,\bar{u}(p')\gamma_\mu u(k)$$

$$\frac{d\sigma^{\text{unpol.}}}{d\Omega} \sim \sum_{\text{final spins}} \overline{\sum_{\text{init. spins}}} |A|^2 \qquad (*)$$

$$\overline{\sum}\left|\frac{e^2}{t}\bar{u}(k')\gamma^\mu u(k)\,\bar{u}(p')\gamma_\mu u(p) - \frac{e^2}{u}\bar{u}(k')\gamma^\mu u(p)\,\bar{u}(p')\gamma_\mu u(k)\right|^2$$

1. First term^2

$$|A-B|^2 = (A-B)^*(A-B)$$

$$= |A|^2 + |B|^2 - \overbrace{2\,\mathrm{Re}(A^*B)}^{A^*B + AB^*}$$

$$\frac{e^4}{t^2}\overline{\sum}\left(\bar{u}(k')\gamma^\mu u(k)\right)^* \bar{u}(k')\gamma^\nu u(k)$$

$$\cdot\left(\bar{u}(p')\gamma_\mu u(p)\right)^* \bar{u}(p')\gamma_\nu u(p)$$

$$*)\ \left(\bar{u}(k')\gamma^\mu u(k)\right)^* = \left(u^+(k')\gamma^0\gamma^\mu u(k)\right)^*$$

$$= u^+(k)\underset{\uparrow\,\gamma^{0\,2}}{}\gamma^{\mu +}\gamma^{0+}u(k') = \bar{u}(k)\gamma^\mu u(k')$$

$$u(k,s) \qquad u(k',s')$$

$$\sum_{s'}\frac{1}{2}\sum_{s}\bar{u}(k,s)\gamma^\mu \underbrace{u(k',s')\bar{u}(k',s')}\gamma^\nu u(k,s)$$

$$\shortparallel$$
$$(\not k' + m)$$

$$= \frac{1}{2}\sum_s \bar{u}_a(k,s)\,\gamma^\mu_{ab}\,(\not k' + m)_{bc}\,\gamma^\nu_{cd}\,u_d(k,s)$$

$$= \frac{1}{2}(\not k + m)_{da}\,\gamma^\mu_{ab}\,(\not k' + m)_{bc}\,\gamma^\nu_{cd}$$

$$= \frac{1}{2}\left[(\not k + m)\gamma^\mu(\not k' + m)\gamma^\nu\right]_{dd} = \underline{\frac{1}{2}\,\mathrm{Tr}\,(\not k + m)\gamma^\mu(\not k' + m)\gamma^\nu}$$

How do we calculate traces of γ-matr.?

$$\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}\begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} = 0$$

Another way: use cyclicity of trace!

$$\mathrm{Tr}(AB\ldots X) = \mathrm{Tr}(B\ldots XA) = \mathrm{Tr}(\ldots XAB)$$

$$\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}(\gamma_5\gamma_5\gamma^\mu) = \mathrm{Tr}(-\gamma_5\gamma^\mu\gamma_5) = -\mathrm{Tr}\,\gamma^\mu$$

cyclicity $\quad = 0$

$$\mathrm{Tr}\,\gamma_5 = \mathrm{Tr}(\gamma^0\gamma^0\gamma_5) = -\mathrm{Tr}\,\gamma_5 = 0$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\{\gamma^\mu,\gamma^\nu\} - \gamma^\nu\gamma^\mu)$$

$$= 2g^{\mu\nu}\,\mathrm{Tr}\,\mathbb{1}_{4\times4} - \underbrace{\mathrm{Tr}\,\gamma^\nu\gamma^\mu}_{=\ \mathrm{Tr}\,\gamma^\mu\gamma^\nu}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma_5\gamma^\mu) = \mathrm{Tr}(-\gamma^\mu\gamma_5) = \mathrm{Tr}(-\gamma_5\gamma^\mu) = 0$$

$$\mathrm{Tr}(\gamma_5 \gamma^\mu \gamma^\nu) = 0$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha) = \mathrm{Tr}\left((\gamma^\beta \gamma_\beta)\gamma^\mu \gamma^\nu \gamma^\alpha\right)$$ Ex.

$$= \emptyset$$

$$\rightarrow \mathrm{Tr.}(\text{odd \# of } \gamma\text{'s}) = 0$$

Useful ID

Ex

$$\gamma^\mu \gamma^\nu \gamma^\alpha = g^{\mu\nu}\gamma^\alpha - g^{\mu\alpha}\gamma^\nu + g^{\nu\alpha}\gamma^\mu + i\varepsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\beta$$

$$\rightarrow \varepsilon_{0123} = +1 = -\varepsilon^{0123}$$

$$\gamma^\mu \gamma^\nu \gamma^\alpha = S^{\mu\nu\alpha\beta}\gamma_\beta + i\varepsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\beta$$

$$S^{\mu\nu\alpha\beta} = g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho) = S^{\mu\nu\alpha\beta}\,\mathrm{Tr}(\gamma_\beta \gamma^\rho) = \underline{4S^{\mu\nu\alpha\rho}}$$

$$\mathrm{Tr}\left[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu\right]$$

$$= \mathrm{Tr}\left[\not{k}'\gamma^\mu\not{k}\gamma^\nu + m^2\gamma^\mu\gamma^\nu + m\left(\cancel{\not{k}'\gamma^\mu\gamma^\nu} + \cancel{\gamma^\mu\not{k}\gamma^\nu}\right)\right]$$

$$= 4\left[k'^\mu k^\nu + k'^\nu k^\mu - (k'k)g^{\mu\nu} + m^2 g^{\mu\nu}\right]$$

$$= 4(k'^\mu k^\nu + k'^\nu k^\mu) + 2t\,g^{\mu\nu}$$

$$\left\{ t = (k-k')^2 = 2m^2 - 2kk' \right.$$

$$\overline{\sum} |A^{(1)}|^2 = \frac{1}{4} \frac{e^4}{t^2} \cdot 4\left[k'^{\mu}k^{\nu} + k'^{\nu}k^{\mu} + \frac{t}{2}g^{\mu\nu}\right]$$

$$\cdot 4\left[p'_{\mu}p_{\nu} + p'_{\nu}p_{\mu} + \frac{t}{2}g_{\mu\nu}\right]$$

$$= \frac{4e^4}{t^2}\Big[2(p'k')(pk) + 2(pk')(p'k) + t^2$$

$$+ t(kk' + pp')\Big]$$

$$(pk) = (p'k') = \frac{s-2m^2}{2}$$

$$(pk') = (p'k) = \frac{2m^2-u}{2}$$

$$(kk') = (pp') = \frac{2m^2-t}{2}$$

$$\Rightarrow \sum |A^{(1)}|^2 = \frac{2e^4}{t^2}\left[(s-2m^2)^2 + (u-2m^2)^2 + m^2 t\right]$$

$|\, 2^{nd}$ term $|^2$

$$\hookrightarrow \left| -\frac{e^2}{u} \bar{u}'(p')\gamma^{\mu}u(k) \; \bar{u}'(k')\gamma_{\mu}u(p)\right|^2$$

$$= \frac{4e^4}{u^2}\left[p'^{\mu}k^{\nu} + p'^{\nu}k^{\mu} + \frac{u}{2}g^{\mu\nu}\right]$$

$$\cdot\left[k'_{\mu}p_{\nu} + k'_{\nu}p_{\mu} + \frac{u}{2}g_{\mu\nu}\right]$$

$p' \leftrightarrow k'$

$t \leftrightarrow u$

$$= \frac{2e^4}{u}\left[(s-2m^2)^2 + (t-2m^2)^2 + m^2 u\right]$$

Cross term

$$-2\frac{e^4}{tu}\sum\left(\bar{u}(k')\gamma^\mu u(k)\,\bar{u}(p')\gamma_\mu u(p)\right)^*$$

$$\cdot\,\bar{u}(k')\gamma^\nu u(p)\,\bar{u}(p')\gamma_\nu u(k)$$

$$=\frac{-2e^4}{tu}\cdot\frac{1}{4}\,\mathrm{Tr}\left[(\not{k}'+m)\underline{\gamma^\mu}(\not{k}+m)\gamma^\nu(\not{p}'+m)\underline{\gamma_\mu}(\not{p}+m)\gamma_\nu\right]$$

Ex

$$\gamma^\mu\gamma^\nu\gamma_\mu=\gamma^\mu(2g^\nu{}_\mu-\gamma_\mu\gamma^\nu)$$

$$=2\gamma^\nu-4\gamma^\nu=-2\gamma^\nu$$

$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu=4g^{\alpha\beta}$$

$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma_\mu=-2\gamma^\gamma\gamma^\beta\gamma^\alpha$$

$\vdots$

$$\gamma^\mu(\not{k}+m)\gamma^\nu(\not{p}'+m)\gamma_\mu=\gamma^\mu\not{k}\gamma^\nu\not{p}'\gamma_\mu+m^2\gamma^\mu\gamma^\nu\gamma_\mu$$

$$+m\gamma^\mu(\not{k}\gamma^\nu+\gamma^\nu\not{p}')\gamma_\mu$$

$$=-2\not{p}'\gamma^\nu\not{k}-2m^2\gamma^\nu+4m(k^\nu+p'^\nu)$$

$$\hookrightarrow\ \mathrm{Tr}(\not{k}'+m)\left(-2\not{p}'\gamma^\nu\not{k}-2m^2\gamma^\nu+4m(k+p')^\nu\right)(\not{p}+m)\gamma_\nu$$

$$=-2\,\mathrm{Tr}(\not{k}'+m)\not{p}'\underbrace{\gamma^\nu\not{k}(\not{p}+m)\gamma_\nu}_{}=4(pk)-2m\not{k}$$

$$-2m^2\,\mathrm{Tr}(\not{k}'+m)\underbrace{\gamma^\nu(\not{p}+m)\gamma_\nu}_{}=4m-2\not{p}$$

$$+4m\,\mathrm{Tr}(\not{k}'+m)(\not{p}+m)(\not{p}'+\not{k})$$

$$=-2\left[4(pk)\cdot4(p'k')-2m^2\,4(p'k)\right]$$

$$-2m^2\left[16m^2 - 2\cdot 4(pk')\right]$$

$$+4m^2 \cdot 4(p'+k, p+k')$$

$$A(k,\underline{s};\, p,\underline{r} \longrightarrow k',\underline{s}';\, p',\underline{r}')$$

Probability for a transition between each spin state

$$\sim \left| A(\text{———}\,''\,\text{———}) \right|^2$$

Total transition probability

$= \sum$ over partial probabilities

# of indep. spin states

$$S \longrightarrow 2S+1$$

$$S = 1/2 \longrightarrow 2S+1 = 2$$

Average over initial spins

$$\hookrightarrow \frac{1}{2}\,\frac{1}{2}$$