Lecture 19

$$S_F^{1/2}(x-y) = i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{\not{p}+m}{p^2-m^2+i\varepsilon}$$

$$\sum_s u^s(p)\,\bar{u}^s(p) = \not{p}+m$$

$$\mathcal{L}_{int} = -\lambda \underset{\text{scalar}}{\phi} \underbrace{\bar{\psi}\psi}_{\text{scalar}}$$

Dimension λ: $\quad S = \int d^4x\, \mathcal{L}$

$[\mathcal{L}] = 4 \qquad [\bar{\psi}(i\not{\partial}-m)\psi] = 4$

$$[\psi] = 3/2$$

$$\left[\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2\right] = 4$$

$$[\phi] = 1$$

$[\mathcal{L}_{int}] = 4 \Rightarrow \quad [\lambda] = 0$

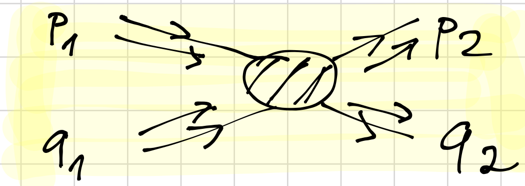
$$\exp[-iH_{int}] \longrightarrow \underline{\sum (-i\lambda)^n \frac{1}{n!}(\phi\bar{\psi}\psi)^n}$$

$H_{int} = -L_{int}$

Time-ordering $\quad T(\psi(x)\bar{\psi}(y)) = \psi(x)\bar{\psi}(y),\ x^0 > y^0$
$\qquad\qquad\qquad\qquad\qquad\quad -\bar{\psi}(y)\psi(x),\ y^0 > x^0$

Normal ordering $\quad :\psi(x_2)\,\psi(x_1): \; = -:\psi(x_1)\,\psi(x_2):$

$$\psi_\alpha(x)\,\bar\psi_\beta(y) = T\left(\psi_\alpha(x)\,\bar\psi_\beta(y)\right) - :\psi_\alpha(x)\,\bar\psi_\beta(y): = S_F^{\alpha\beta}(x-y)$$



$\underline{NN \to NN}$

1. $|i\rangle = \underline{\sqrt{2E_{p_1}}\,\sqrt{2E_{q_1}}}\; b_{p_1}^{s_1 +}\, b_{q_1}^{t_1 +}\,|0\rangle$

2. $\langle f| = \sqrt{2E_{p_2}}\,\sqrt{2E_{q_2}}\,\langle 0|\, b_{q_2}^{t_2}\, b_{p_2}^{s_2}$

At order λ^2

$$S_{fi}^{(2)} = \frac{(-i\lambda)^2}{2!}\int d^4x_1\, d^4x_2\; T\Big[\bar\psi(x_1)\psi(x_1)\phi(x_1)\cdot \bar\psi(x_2)\psi(x_2)\phi(x_2)\Big]$$

$$\hookrightarrow\; :\bar\psi(x_1)\psi(x_1)\bar\psi(x_2)\psi(x_2):\;\phi(x_1)\,\phi(x_2)$$

$$\phi(x_1)\,\phi(x_2) = i\int \frac{d^4q}{(2\pi)^4}\,\frac{e^{-iq(x_1-x_2)}}{q^2-\mu^2+i\varepsilon}$$

$$:\bar\psi(x_1)\psi(x_1)\bar\psi(x_2)\psi(x_2):\;|i\rangle$$

$NN \to NN \longrightarrow$ we only need b's acting on $|i\rangle$

— " — b^+ — " — $|f\rangle$

$$\psi(x_1) = \int \frac{d^3k_1}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{k_1}}}\; b_{k_1}^{r}\, u_r(k_1)\, e^{-ik_1x_1}$$

$$\hookrightarrow\; -\int \frac{d^3k_1\, d^3k_2}{(2\pi)^6}\,\left[\bar\psi(x_1)\cdot u_{r_1}(k_1)\right]\left[\bar\psi(x_2)\cdot u_{r_2}(k_2)\right]$$

$$\underline{\sqrt{4E_{p_1}E_{q_1}}}\cdot \frac{e^{-ik_1x_1-ik_2x_2}}{\underline{\sqrt{4E_{k_1}E_{k_2}}}}\; b_{k_1}^{r_1}\, b_{k_2}^{r_2}\, b_{p_1}^{s_1 +}\, b_{q_1}^{t_1 +}\,|0\rangle$$

$$-\Big\{\underline{[\bar\Psi(x_1)\cdot u_{t_1}(q_1)][\bar\Psi(x_2)\cdot u_{s_1}(p_1)]}\, e^{-ip_1x_2-iq_1x_1}$$

$$-\underline{[\bar\Psi(x_1)\; u_{s_1}(p_1)][\bar\Psi(x_2)\cdot u_{t_1}(q_1)]}\, e^{-ip_1x_1-iq_1x_2}\Big\}|0\rangle$$

$$\langle f| - \{ \cdots \} |0\rangle$$

$$\langle 0|\, b_{q_2}^{t_2}\, b_{p_2}^{s_2} \int \frac{d^3k_1\, d^3k_2}{(2\pi)^6} \Big[b_{k_1}^{r_1+}\, \bar u_{r_1}(k_1)\, e^{+ik_1x_1}\, u_{t_1}(q_1)$$

$$\cdots$$

$$\Downarrow \qquad \langle 0|0\rangle = 1$$

$$\bar u_{s_2}(p_2)\, u_{t_1}(q_1)\;\; \bar u_{t_2}(q_2)\, u_{s_1}(p_1)\; e^{ip_2x_1+iq_2x_2}$$

$$-\;\bar u_{t_2}(q_2)\, u_{t_1}(q_1)\;\bar u_{s_2}(p_2)\, u_{s_1}(p_1)\, e^{ip_2x_2+iq_2x_1}$$

$$S_{fi}^{(2)} = (-i\lambda)^2 \int \frac{d^4x_1\, d^4x_2\, d^4k}{(2\pi)^4}\; \frac{i\, e^{ik(x_1-x_2)}}{k^2-m^2+i\varepsilon}$$

$$\cdot\Big\{ \bar u_{s_2}(p_2)\, u_{s_1}(p_1)\; \bar u_{t_2}(q_2)\, u_{t_1}(q_1)\; e^{ix_1(q_2-q_1)+ix_2(p_2-p_1)}$$

$$-\; \bar u(p_2)\, u(q_1)\; \bar u(q_2)\, u(p_1)\, e^{ix_1(p_2-q_1)+ix_2(q_2-p_1)}\Big\}$$

$$\int dx_1\, dx_2 \cdots (2\pi)^8\, \delta^4(k+q_2-q_1)\; \delta^4(p_2-p_1-k)$$

$$(\quad)(\quad)$$

$$-(2\pi)^8\, \delta^4(k+p_2-q_1)\; \delta^4(q_2-p_1-k)$$

$$(\quad)\quad(\quad)$$

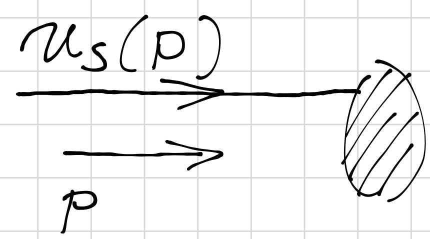
$$S^{(2)}_{fi} = \delta_{fi} + i A^{(2)}_{fi} (2\pi)^4 \delta^4 (p_1 + q_1 - p_2 - q_2)$$

$$A^{(2)}_{fi} = (-i\lambda)^2 \left[\frac{\bar{u}(p_2)\, u(p_1)\, \bar{u}(q_2)\, u(q_1)}{(p_2 - p_1)^2 - m^2 + i\varepsilon} - \frac{\bar{u}(p_2)\, u(q_1)\, \bar{u}(q_2)\, u(p_1)}{(q_2 - p_1)^2 - m^2 + i\varepsilon} \right]$$

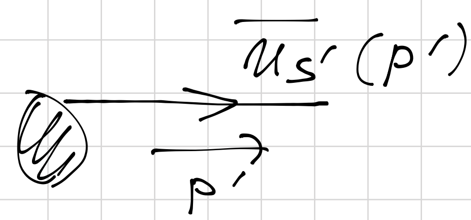
(p_1, s_1) (p_2, s_2) (q_1, t_1) (q_2, t_2)

Feynman rules

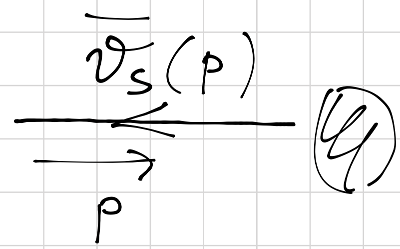
*) Incoming fermion
write $u_s(\vec{p})$



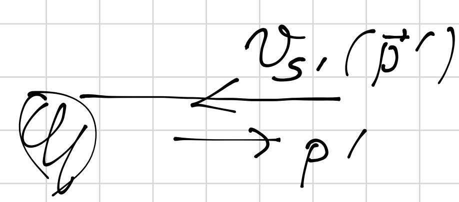
*) Outgoing fermion
write $\bar{u}_{s'}(\vec{p}\,')$



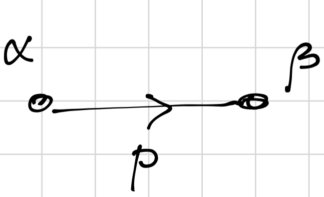
*) Incoming anti-p.
write $\bar{v}_s(\vec{p})$



*) Outgoing anti-part.
write $v_{s'}(\vec{p}\,')$



*) Internal fermion lines



$$\frac{i(\not{p} + M)^{\alpha\beta}}{p^2 - M^2 + i\varepsilon}$$

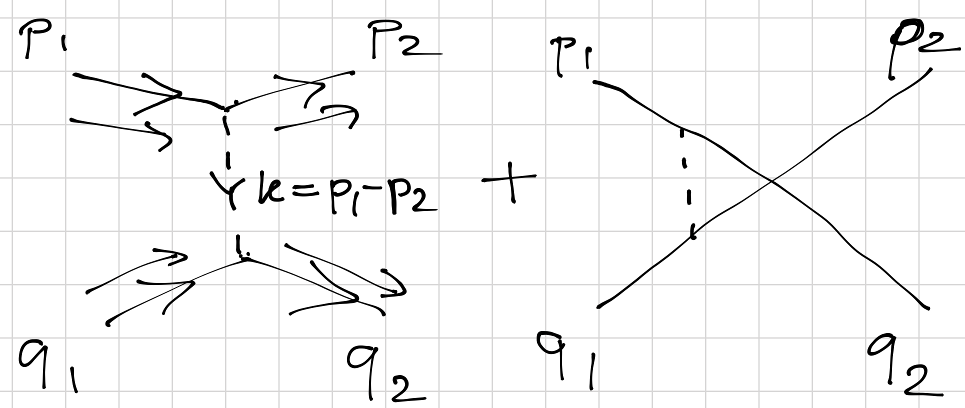
*) Closed fermion loop $\longrightarrow$ "$-$" sign

*) Scalar int. $(-i\lambda)$ at each vertex

*) Internal scalar line $\dfrac{i}{p^2 - m^2 + i\varepsilon}$

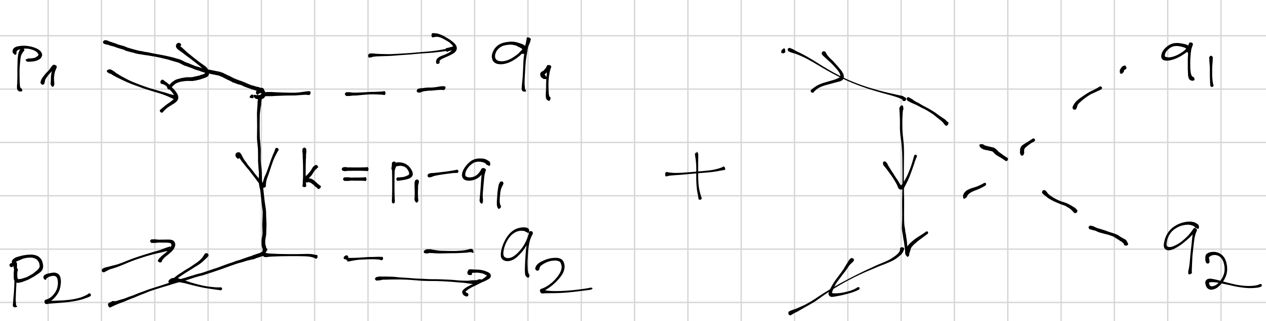
- Impose momentum conservation at each vertex
- Integrate over loop momenta
- Add "−" sign for Fermi statistics

$A^{(2)}_{fi}\,(NN \to NN)$



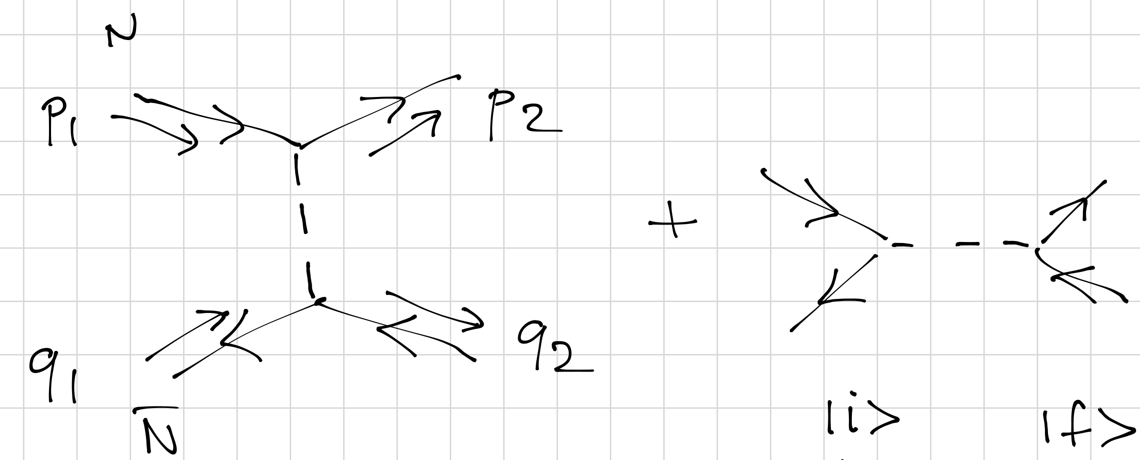
$$(-i\lambda)^2 \left[\frac{\bar u(p_2)\, u(p_1)\;\; \bar u(q_2)\, u(q_1)}{(p_1 - p_2)^2 - m^2 + i\varepsilon} - \frac{\bar u(q_2)\, u(p_1)\;\; \bar u(p_2)\, u(q_1)}{(p_1 - q_2)^2 - m^2 + i\varepsilon} \right]$$

$A^{(2)}_{fi}\,(N\bar N \to \phi\phi)$



$$(-i\lambda)^2 \left[\frac{\bar v(p_2)(\not p_1 - \not q_1 + M)\, u(p_1)}{(p_1 - q_1)^2 - M^2 + i\varepsilon} + \frac{\bar v(p_2)(\not p_1 - \not q_2 + M)\, u(p_1)}{(p_1 - q_2)^2 - M^2 + i\varepsilon} \right]$$

$N\bar{N} \longrightarrow N\bar{N}$



$|i\rangle$ $|f\rangle$

$$A^{(2)}_{fi} = (-i\lambda)^2 \left[- \frac{\bar{u}(p_2)\, u(p_1)\, \bar{v}(q_1)\, v(q_2)}{(p_1 - p_2)^2 - M^2 + i\varepsilon} + \frac{\bar{v}(q_1)\, u(p_1)\, \bar{u}(p_2)\, v(q_2)}{(p_1 + q_1)^2 - m^2 + i\varepsilon} \right]$$

To go from QFT $\longrightarrow$ NR potential scatt

$$u = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\, \xi \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi \end{pmatrix} \longrightarrow \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$v \longrightarrow \sqrt{m} \begin{pmatrix} \xi \\ -\xi \end{pmatrix}$$

$$\bar{u} u = 2m\, \delta_{ss'} = -\bar{v} v$$

$$(p_1 - p_2)^2 = -(\vec{p}_1 - \vec{p}_2)^2 = -\vec{\Delta}^2$$

$$NN: \quad U(r) = -\int \frac{d^3 \vec{\Delta}}{(2\pi)^3} A \frac{e^{i \vec{\Delta} \vec{r}}}{(2m)^2} = -\lambda^2 \int \frac{d^3 \vec{\Delta}}{(2\pi)^3} \frac{e^{i \vec{\Delta} \vec{r}}}{\vec{\Delta}^2 + m^2}$$

$$= -\frac{\lambda^2}{4\pi} \frac{e^{-mr}}{r}$$

$$N\bar{N}: \quad U(r) = -\frac{\lambda^2}{4\pi} \frac{e^{-mr}}{r} \delta_{ss'} \delta_{rr'}$$

Pseudoscalar interaction

$$\mathcal{L}_{int} = -\lambda \phi \bar{\Psi} \gamma_5 \Psi$$

$$\uparrow$$

$$PS: P: \phi(\vec{x}, t) \longrightarrow -\phi(-\vec{x}, t)$$

$NN \to NN$

$$A_{fi}^{(2)} = (-i\lambda)^2 \left[\frac{\bar{u}(p_2)\gamma_5 u(p_1) \; \bar{u}(q_2)\gamma_5 u(q_1)}{(p_1-p_2)^2 - m^2 + i\varepsilon} - (p_2 \leftrightarrow q_2) \right]$$

What is the NR limit?

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$u \longrightarrow \sqrt{M}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad v \longrightarrow \sqrt{M}\begin{pmatrix} \xi \\ -\xi \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\bar{u}\gamma_5 u \longrightarrow 0$$

$$(p\sigma) = \overset{\sqrt{M^2+\vec{p}^2}}{E}\mathbb{1} - \vec{p}\vec{\sigma}$$

$$\approx M\mathbb{1} - \vec{p}\vec{\sigma}$$

$$u = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\,\xi \\ \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi \end{pmatrix}\Bigg|_{m \gg |\vec{p}|}$$

$$\sqrt{p\sigma} \approx \sqrt{M} - \frac{\vec{p}\vec{\sigma}}{2\sqrt{M}}$$

$$u \simeq \sqrt{M}\begin{pmatrix} \xi \\ \xi \end{pmatrix} + \frac{1}{2\sqrt{M}}\begin{pmatrix} -\vec{p}\vec{\sigma}\,\xi \\ +\vec{p}\vec{\sigma}\,\xi \end{pmatrix}$$

$$\sqrt{p\bar{\sigma}} \approx \sqrt{M} + \frac{\vec{p}\vec{\sigma}}{2\sqrt{M}}$$

$$\bar{u}(p_2)\gamma_5 u(p_1) = \begin{pmatrix} \sqrt{p_2\sigma}\;\xi \\ \sqrt{p_2\bar{\sigma}}\;\xi \end{pmatrix}^{+} \gamma^0\gamma_5 \begin{pmatrix} \sqrt{p_1\sigma}\;\xi \\ \sqrt{p_1\bar{\sigma}}\;\xi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{M}\begin{pmatrix} \xi \\ \xi \end{pmatrix} + \frac{1}{2\sqrt{M}}\begin{pmatrix} (\vec{p}'\vec{\sigma})\,\xi \\ -(\vec{p}'\vec{\sigma})\,\xi \end{pmatrix} \end{pmatrix}^{+} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\sqrt{M}\begin{pmatrix} \xi \\ \xi \end{pmatrix} + \frac{1}{2\sqrt{M}}\begin{pmatrix} -\vec{p}\vec{\sigma}\;\xi \\ \vec{p}\vec{\sigma}\;\xi \end{pmatrix} \right)$$

$$= \xi^{+}\left[(\vec{p}'\vec{\sigma}) - (\vec{p}\vec{\sigma})\right]\xi$$

$$\underline{\vec{p} - \vec{p}\,' \neq 0}$$

$$-\lambda^2 \frac{\xi^{+}_{s'}(\vec{p}_1 - \vec{p}_2, \vec{\sigma})\,\xi_s \quad \xi^{+}_{r'}(\vec{p}_1 - \vec{p}_2, \vec{\sigma})\,\xi_r}{(\vec{p}_1 - \vec{p}_2)^2 + m^2}$$

$$\downarrow$$

$$U(r) = -\int \frac{d^3\vec{\Delta}}{(2\pi)^3} \frac{\langle \vec{\Delta}\vec{\sigma}\rangle_{ss'} \langle \vec{\Delta}\vec{\sigma}\rangle_{rr'}}{\vec{\Delta}^2 + m^2} \cdot \frac{-\lambda^2}{(2M)^2} e^{i\vec{\Delta}\vec{r}}$$

$$\sim \frac{\vec{\Delta}^2}{4M^2} \longrightarrow \text{recoil}$$