Lecture 18

Spinors in Weyl representation

Products of spinors

"Inner"
$$\bar{u}_{s'}(\vec{p})\, u_s(\vec{p}) = 2\delta_{ss'} \cdot m$$
$$\bar{v}_{s'}(\vec{p})\, v_s(\vec{p}) = -2m\,\delta_{ss'}$$
$$\bar{u}_{s'}(\vec{p})\,\gamma^0\, u_s(\vec{p}) = 2E_p\,\delta_{ss'}$$
$$\bar{v}_{s'}(\vec{p})\,\gamma^0\, v_s(\vec{p}) = -2E_p\,\delta_{ss'}$$

Ex

Weyl, Dirac
$$\bar{u}_{s'}(\vec{p})\, v_s(\vec{p}) = 0$$
$$\bar{u}_{s'}(\vec{p})\,\gamma^0\, v_s(-\vec{p}) = 0$$

$\vdots$

"Outer"
$$\sum_s u_s(\vec{p})\,\bar{u}_s(\vec{p}) = \not{p} + m$$
$$\sum_s v_s(\vec{p})\,\bar{v}_s(\vec{p}) = \not{p} - m$$

$$\{\psi(x), \psi(y)\} = \{\pi(x), \pi(y)\} = 0 \qquad \pi = i\psi^+$$
$$\{\psi_\alpha(\vec{x}), \psi^+_\beta(\vec{y})\} = \delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$$

$$\psi(\vec{x}) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Big[b^s_{\vec{p}}\,\underline{u_s(\vec{p})}\,e^{i\vec{p}\vec{x}} + c^{s+}_{\vec{p}}\,\underline{v_s(\vec{p})}\,e^{-i\vec{p}\vec{x}}\Big]$$

$$\psi^+(\vec{y}) = \sum_{s'} \int \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\Big[b^{s'+}_{\vec{q}}\,u^+_{s'}(\vec{q})\,e^{-i\vec{q}\vec{y}} + c^{s'}_{\vec{q}}\,v^+_{s'}(\vec{q})\,e^{i\vec{q}\vec{y}}\Big]$$

$$\{ b^s_{\vec p}, b^{s'+}_{\vec q} \} = \{ c^s_{\vec p}, c^{s'+}_{\vec q} \} = \underline{\delta_{ss'} (2\pi)^3 \delta^3(\vec p - \vec q)}$$

All other $\{\ \} = 0$

$$\{ \psi_\alpha(\vec x), \psi^+_\beta(\vec y) \} = \sum_{s,s'} \int \frac{d^3\vec p\, d^3\vec q}{(2\pi)^6} \frac{1}{\sqrt{4E_p E_q}}$$

$$\times \Big[\{ b^s_{\vec p}, b^{s'+}_{\vec q} \} u_s(\vec p)\, u^+_{s'}(\vec q)\, e^{i(\vec p \vec x - \vec q \vec y)}$$

$$+ \underbrace{\{ c^{s+}_{\vec p}, c^{s'}_{\vec q} \}}_{\delta_{ss'}(2\pi)^3 \delta^3(\vec p - \vec q)} v_s(\vec p)\, v^+_{s'}(\vec q)\, e^{-i(\vec p \vec x - \vec q \vec y)} \Big]$$

$$= \sum_s \int \frac{d^3\vec p}{(2\pi)^3} \frac{1}{2E_p} \Big[u_s(\vec p)\, \bar u_s(\vec p)\, \gamma^0 e^{i\vec p(\vec x - \vec y)}$$

$$+ v_s(\vec p)\, \bar v_s(\vec p)\, \gamma^0 e^{-i\vec p(\vec x - \vec y)} \Big]$$

$$= \underbrace{\int \frac{d^3\vec p}{(2\pi)^3} \frac{1}{2E_p}} \Big\{ (\not p + m)\gamma^0 e^{i\vec p(\vec x - \vec y)} + (\not p - m)\gamma^0 e^{-i\vec p(\vec x - \vec y)} \Big\}$$

$\not p = \gamma^0 E_p - \vec\gamma \vec p$

$$(E_p + m\gamma^0)\, e^{i\vec p(\vec x - \vec y)} + (E_p - m\gamma^0)\, e^{-i\vec p(\vec x - \vec y)}$$

$$- \vec\gamma \gamma^0 \vec p \left(e^{i\vec p(\vec x - \vec y)} - e^{-i\vec p(\vec x - \vec y)} \right)$$

$\underline{\text{even u. } \vec p \to -\vec p}$

$$= \int \frac{d^3\vec p}{(2\pi)^3} \frac{1}{2E_p}\, 2E_p\, e^{i\vec p(\vec x - \vec y)} = \delta^3(\vec x - \vec y)$$

$\underline{\text{OK}}$

$$H = \pi \dot\psi - \mathcal{L} \qquad \pi = i\psi^+ = i\bar\psi\gamma^0$$

$$= \bar\psi\, i\partial^0 \gamma_0\, \psi - \bar\psi(i\not\partial - m)\psi = \bar\psi\, \underline{(-i\gamma^i \partial_i + m)}\, \psi$$

$$\not\partial = \partial^0\gamma_0 + \underline{\partial^i\gamma_i} \qquad \underline{g^i{}_i = -1}$$

$$\longrightarrow \quad \partial_i = \frac{\partial}{\partial x^i} \qquad \partial_i e^{i\vec{p}\vec{x}} = \frac{\partial}{\partial x^i} e^{-ip_i x^i} = \underline{-ip_i\, e^{i\vec{p}\vec{x}}}$$

$$\partial_i x^i \neq \partial_i x_i \quad (\text{opposite sign})$$

To derive $H \longrightarrow$ consider $(-i\gamma^i\partial_i + m)\psi(x)$

$$(-i\gamma^i\partial_i + m)\psi(\vec{x}) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}$$

$$\times \Big\{ b_p^s \underbrace{(-i\gamma^i\partial_i + m)\, u_s(\vec{p})\, e^{i\vec{p}\vec{x}}}_{=(-\gamma^i p_i + m)u_s(\vec{p})\, e^{i\vec{p}\vec{x}} \;=\; \gamma^0 E_p\, u_s(\vec{p})\, e^{i\vec{p}\vec{x}}}$$

$$+ c_p^{s+} \underbrace{(-i\gamma^i\partial_i + m)\, v_s(\vec{p})\, e^{-i\vec{p}\vec{x}}}_{=(+\gamma^i p_i + m)v_s(\vec{p})\, e^{-i\vec{p}\vec{x}} \;=\; -\gamma^0 E_p\, v_s(\vec{p})\, e^{-i\vec{p}\vec{x}}} \Big\}$$

$$(\not p - m)\, u_s(\vec{p}) = 0 = (\gamma^0 p_0 + \gamma^i p_i - m)u = 0$$

$$(-\gamma^i p_i + m)u = \gamma^0 p^0 u$$

$$(\not p + m)\, v_s(\vec{p}) = 0 \rightsquigarrow (\gamma^i p_i + m)v = -\gamma^0 p^0 v$$

$$\downarrow$$

$$(-i\gamma^i\partial_i + m)\psi(\vec{x}) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\,\gamma^0$$

$$\left[b_p^s\, u_s(\vec{p})\, e^{i\vec{p}\vec{x}} - c_p^{s+}\, v_s(\vec{p})\, e^{-i\vec{p}\vec{x}} \right]$$

$$H = \int d^3\vec{x}\; \psi^+(\vec{x})\,\gamma^0 (-i\gamma^i\partial_i + m)\,\psi(\vec{x})$$

$$= \sum_{s,s'} \int d^3\vec{x} \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6}\sqrt{\frac{E_p}{4E_q}}$$

$$\cdot\left[b_q^{s'\dagger} u_{s'}^\dagger(\vec{q})\, e^{-i\vec{q}\vec{x}} + c_q^{s'} v_{s'}^\dagger(\vec{q})\, e^{i\vec{q}\vec{x}}\right]$$

$$\left[b_p^{s} u_s(\vec{p})\, e^{i\vec{p}\vec{x}} - c_p^{s\dagger} v_s(\vec{p})\, e^{-i\vec{p}\vec{x}}\right]$$

$$\int d^3\vec{x}\; e^{i\vec{p}\vec{x} \pm i\vec{q}\vec{x}} = (2\pi)^3\, \delta^3(\vec{p} \pm \vec{q})$$

$$= \sum_{s,s'} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2} \Big\{ b_p^{s'\dagger} b_p^{s} \underbrace{u_{s'}^\dagger(\vec{p})\, u_s(\vec{p})}_{= 2E_p \delta_{ss'}}$$

$$- c_p^{s'} c_p^{s\dagger} \underbrace{v_{s'}^\dagger(\vec{p})\, v_s(\vec{p})}_{= 2E_p \delta_{ss'}}$$

$$- b_{-p}^{s'\dagger} c_p^{s\dagger} \underbrace{u_{s'}^\dagger(-\vec{p})\, v_s(\vec{p})}_{= 0}$$

$$+ c_{-p}^{s'} b_p^{s} \underbrace{v_{s'}^\dagger(-\vec{p})\, u_s(\vec{p})}_{= 0} \Big\}$$

$$= \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} E_p \left[b_p^{s\dagger} b_p^{s} - c_p^{s} c_p^{s\dagger} \right]$$

$$= \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \underline{E_p} \Big[b_p^{s\dagger} b_p^{s} + c_p^{s\dagger} c_p^{s} \underbrace{- (2\pi)^3 \delta^3(0)}_{\to \text{ negative } E \text{ of vacuum}} \Big]$$

We diagonalized the H

Define vacuum $|0\rangle$

$$c_p^s |0\rangle = b_p^s |0\rangle = 0$$

1 part. $|\vec{p}, s\rangle = \sqrt{2E_p}\; b_p^{s+}|0\rangle$

2 part. $|p_1, s_1; p_2, s_2\rangle = \sqrt{4E_{p_1}E_{p_2}}\; b_{p_1}^{s_1+} b_{p_2}^{s_2+}|0\rangle$
$= -|p_2, s_2; p_1, s_1\rangle$

Pauli exclusion: $\underline{|p,s;p,s\rangle = -|p,s;p,s\rangle}$

Commutation relations $[H, b_{\dots}]$

$[H, b_p] = -E_p b_p \qquad [H, c_p] = -E_p c_p$

Ex

$$[b_p^{s+} b_p^s + c_p^{s+} c_p^s, b_q^{s'}]$$

$$= b_p^{s+} b_p^s b_q^{s'} - b_q^{s'} b_p^{s+} b_p^s$$

$$+ \cancel{c_p^{s+} c_p^s b_q^{s'}} - \cancel{b_q^{s'} \underbrace{c_p^{s+}}_{``-"} \underbrace{c_p^s}_{``-"}}$$

$$= \cancel{b_p^{s+} b_p^s b_q^{s'}} - \underline{(2\pi)^3 \delta^3(\vec{p}-\vec{q})\, \delta_{ss'} b_p^s} + \cancel{b_p^{s+} \underbrace{b_q^{s'} b_p^s}_{``-"}}$$

$$\Downarrow$$

$$[H, b_q^{s'}] = -E_q\, b_q^{s'}$$

Ex.

$\psi(\vec{x}, t)$: $\dfrac{\partial \psi}{\partial t} = i[H, \psi]$

$$\psi(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b_p^s u_s(p) e^{-ipx} + c_p^{s+} v_s(p) e^{ipx} \right]$$

$$\bar{\Psi}(y) = \sum_{s'} \int \frac{d^3\bar{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \Big[b_q^{s'+} \bar{u}_{s'}(q) e^{iqy} + c_q^{s'} \bar{v}_{s'}(q) e^{-iqy} \Big]$$

We want to derive Feynman propagator for spinor field (fermion)

$$\langle 0 | \Psi_\alpha(x) \bar{\Psi}_\beta(y) | 0 \rangle = \underbrace{\sum_{ss'}} \int \frac{d^3\bar{p}\, d^3\bar{q}}{(2\pi)^6} \frac{1}{\sqrt{4E_p E_q}}$$

$$\times \underbrace{u_s^\alpha(\bar{p}) \bar{u}_{s'}^\beta(\bar{q})}_{(\not{p}+m)^{\alpha\beta}} e^{-ipx+iqy} \langle 0 | b_p^s b_q^{s'+} | 0 \rangle$$

$$b_p^s b_q^{s'+} + b_q^{s'+} b_p^s = \{ \} = \delta_{ss'} (2\pi)^3 \delta^3(\bar{p}-\bar{q})$$

$$\{ (\not{p}+m) e^{-ip(x-y)} = (i\not{\partial}_x + m) e^{\cdots}$$

$$= (i\not{\partial}_x + m) \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

$$\langle 0 | \bar{\Psi}_\beta(y) \Psi_\alpha(x) | 0 \rangle = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\bar{v}_s^\beta v_s^\alpha}_{\not{p}-m} e^{ip(x-y)}$$

$$\langle 0 | \{ c_q^{s'}, c_p^{s+} \} | 0 \rangle = \delta_{ss'} (2\pi)^3 \delta^3(p-q)$$

$$\bar{v}_s v = \bar{v}_s^\alpha v_s^\alpha$$

$$\hookrightarrow \langle 0 | \bar{\Psi}_\beta(y) \Psi_\alpha(x) | 0 \rangle = -(i\not{\partial}_x + m) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{ip(x-y)}$$

$$\langle 0 | \{ \Psi, \bar{\Psi} \} | 0 \rangle = i S(x-y)$$

$$= (i\not{\partial}_x + m) \big[D(x-y) - D(y-x) \big]$$

$$D(x-y) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

$$(-i\partial\!\!\!/_x + m)\; iS(x-y) = (\Box_x + m^2)\left[D(x-y) - D(y-x)\right]$$

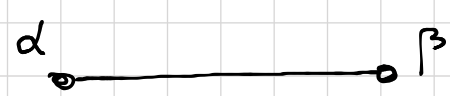
$$= 0 \quad \text{(away from sing.)}$$

Define Feynman prop.

$$S_F(x-y) = \langle 0 | T\,\psi(x)\bar{\psi}(y) | 0 \rangle$$

$$= \begin{cases} \langle 0 | \psi(x)\bar{\psi}(y) | 0 \rangle, & x^0 > y^0 \\ -\langle 0 | \bar{\psi}(y)\psi(x) | 0 \rangle, & y^0 > x^0 \end{cases}$$

$$S_F^{\alpha\beta}(x-y) = i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{(p\!\!\!/ + m)^{\alpha\beta}}{p^2 - m^2 + i\varepsilon}$$



$$\boxed{(i\partial\!\!\!/_x - m)\, S_F(x-y) = i\,\delta^{(4)}(x-y)}$$

S_F — Green's function of Dirac eq.

Ex (?)

Modification of Wick's theorem

$$:\psi_2\psi_1: \; = \; -:\psi_1\psi_2: \quad \text{normal ordering}$$

Contraction

$$\overbrace{\psi_\alpha(x)\,\bar{\psi}_\beta(y)} = T\,\psi(x)\bar{\psi}(y) - :\psi(x)\bar{\psi}(y):$$
$$= S_F^{\alpha\beta}(x-y)$$

{ I will not rederive Wick's theorem anew

Interacting fermions

a). Fermions + scalar

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - M)\psi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$
$$-\lambda\phi\bar{\psi}\psi$$

Consider $\psi\psi \to \psi\psi$

$$|i\rangle = \sqrt{4E_{p_1}E_{q_1}}\; b_{p_1}^{s_1\,+}\, b_{q_1}^{t_1\,+}\,|0\rangle = |p_1 s_1; q_1 t_1\rangle$$

$$\langle f| = \sqrt{4E_{p_2}E_{q_2}}\;\langle 0|\, b_{q_2}^{t_2}\, b_{p_2}^{s_2}$$

$$|f\rangle = |i\rangle\,(p_1, s_1 \to p_2, s_2\,;\; q_1, t_1 \to q_2, t_2)$$

At order λ^2

$$S_{fi} = \frac{(-i\lambda)^2}{2}\int d^4x_1\, d^4x_2$$
$$T\left(\bar{\psi}(x_1)\psi(x_1)\phi(x_1)\bar{\psi}(x_2)\psi(x_2)\phi(x_2)\right)$$

$$:\bar{\psi}\psi\bar{\psi}\psi:\;\phi\,\phi$$