Lecture 17

Dirac eq;
{Weyl, Majorana (Dirac) represent. of the Clifford algebra}
$\mathcal{P}, C$ transf.
L, R projectors $\frac{1}{2}(1 \mp \gamma_5)$

Symmetries of Dirac eq.;
Conserved currents
Plane wave solutions
Quntization

$$\mathcal{L} = \bar{\psi}\,\underline{(i\not{\partial} - m)}\,\psi \qquad \text{Slash } \not{\partial} = \partial_\mu \gamma^\mu$$

1. Translations $x \longrightarrow x + \varepsilon$

$$\psi \longrightarrow \psi(x+\varepsilon) = \psi(x) + \underbrace{\varepsilon^\mu \partial_\mu \psi(x)}_{\delta\psi}$$

$$\delta\mathcal{L} = \underline{\varepsilon^\mu \partial_\mu \mathcal{L}}$$

Noether current

$$\partial_\mu \left(\underbrace{\frac{\partial \mathcal{L}}{\partial_\mu \psi}}_{= i\bar{\psi}\gamma^\mu} \underbrace{\delta\psi}_{= \varepsilon^\alpha \partial_\alpha \psi} - g^{\mu\nu} \varepsilon_\nu \mathcal{L} \right) = 0$$

$$*) \quad \boxed{T^{\mu\nu} = \bar{\psi}\, i\gamma^\mu \partial^\nu \psi - \underbrace{g^{\mu\nu}\mathcal{L}}_{\to \text{E.o.M.}}}$$

*) Total energy

$$E = \int d^3\vec{x}\, T^{00} = \int d^3\vec{x}\, \bar{\Psi}\, \underbrace{i\gamma^0\partial^0}\, \Psi$$

$$\partial\!\!\!/ = \gamma^0\partial_0 + \gamma^i\partial_i \qquad (i\partial\!\!\!/ - m)\Psi = 0$$

$$(i\partial^0\gamma^0 + i\partial^j\gamma_j - m)\Psi = 0$$

$$\underline{E = \int d^3\vec{x}\, \bar{\Psi}(-i\gamma^i\partial_i + m)\Psi}$$

2. Lorentz transf.

$$\Lambda = \exp\left[\tfrac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right] \qquad \ll g_{\rho\sigma}$$

$$S[\Lambda] = \exp\left[\tfrac{1}{2}\Omega_{\rho\sigma} S^{\rho\sigma}\right]$$

$$\Psi(x) \to S[\Lambda]\,\Psi(\Lambda^{-1}x)$$

$$\delta\Psi^\alpha = \tfrac{1}{2}\Omega_{\rho\sigma}[S^{\rho\sigma}]^\alpha{}_\beta \Psi^\beta - \tfrac{1}{2}\Omega_{\rho\sigma}[M^{\rho\sigma}]^{\mu\nu} x_\nu \partial_\mu \Psi^\alpha$$

$$[M^{\rho\sigma}]^{\mu\nu} = g^{\rho\mu}g^{\sigma\nu} - g^{\rho\nu}g^{\sigma\mu}$$

$$\hookrightarrow \delta\Psi^\alpha = \tfrac{1}{2}\Omega_{\rho\sigma}\left[\underline{[S^{\rho\sigma}]^{\alpha\beta}\Psi_\beta} - (x^\rho\partial^\sigma - x^\sigma\partial^\rho)\Psi^\alpha\right]$$

$$J^{\mu\rho\sigma} = \underbrace{x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}}_{\text{was there for scalar}} - \underbrace{i\bar{\Psi}\gamma^\mu S^{\rho\sigma}\Psi}_{\text{New term}}$$

3. Internal vector symmetry

$$\Psi \to e^{-i\alpha}\Psi \to \text{conserved current}$$

$$j_V^\mu = \bar{\Psi}\gamma^\mu\Psi$$

Ex (?) $\quad \partial_\mu j_V^\mu = 0$ $\qquad$ E.o.M. $\quad (i\partial\!\!\!/ - m)\Psi = 0$

$$\bar{\Psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0$$

4 Axial symmetry

$$\psi \to e^{i\alpha\gamma_5}\psi \qquad \bar\psi \to \bar\psi e^{i\alpha\gamma_5}$$

$$\underline{e^{i\alpha\gamma_5}\gamma^0 = \gamma^0 e^{-i\alpha\gamma_5}}$$

$$j_A^\mu = \bar\psi\gamma^\mu\gamma_5\psi \qquad \text{conserved only for } m=0$$

$$\partial_\mu j_A^\mu = 2i\underline{\underline{m}}\,\bar\psi\gamma_5\psi \qquad \text{Ex. (?)}$$

Plane wave solutions

$$(i\not\partial - m)\psi = 0$$

a. Positive energy $\psi = u(\vec p)\, e^{-ipx}$

$$(i\not\partial - m)e^{-ipx}u(\vec p) = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar\sigma^\mu & -m \end{pmatrix} u\, e^{-ipx} = 0$$

$$\sigma^\mu = (1, \vec\sigma) \qquad \vec\gamma = \begin{pmatrix} 0 & \vec\sigma \\ -\vec\sigma & 0 \end{pmatrix}$$
$$\bar\sigma^\mu = (1, -\vec\sigma)$$

$\underline{\text{Solution}}$ $$u(\vec p) = \begin{pmatrix} \sqrt{(p\sigma)}\,\xi_{2\times 2} \\ \sqrt{(p\bar\sigma)}\,\xi \end{pmatrix}$$

ID: $$(p\sigma)(p\bar\sigma) = (E_p - \vec p\vec\sigma)(E_p + \vec p\vec\sigma) \qquad \text{Ex. (?)}$$
$$= E^2 - \vec p^2 = m^2$$

ξ 2-spinor choose $\underline{\xi^\dagger\xi = 1}$

Negative energy solution $\psi = v(\vec p)\,e^{+ipx}$

$$(\not p + m)v(\vec p) = 0 \longrightarrow v(\vec p) = \begin{pmatrix} \sqrt{(p\sigma)}\,\eta \\ -\sqrt{(p\bar\sigma)}\,\eta \end{pmatrix}$$

$$\eta_{2\times 2} \; : \; \eta^{\dagger}\eta = 1$$

Consider pos-E solution at rest

$p^{\mu} = (m, \vec{0})$ $\qquad$ $\underline{u(\vec{p}) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}}$

ξ describes the spin state

Chose spin along z direction $\xi_{\pm 1}$

$$\sigma^3 \xi_{\pm 1} = \pm \xi_{\pm 1} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\xi_{\uparrow} \quad \xi_{\downarrow} \qquad \xi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let's boost in the z-direction

$$p^{\mu} = (E, 0, 0, p)$$

$$u_{\uparrow} = \begin{pmatrix} \sqrt{(p\sigma)}\,\xi_{\uparrow} \\ \sqrt{(p\bar{\sigma})}\,\xi_{\uparrow} \end{pmatrix} = \begin{pmatrix} \sqrt{E-p}\,\xi_{\uparrow} \\ \sqrt{E+p}\,\xi_{\uparrow} \end{pmatrix}$$

For massless particle $\quad m=0 \quad \underline{E=p}$

$$\underline{\underline{u_{\uparrow}}} = \begin{pmatrix} 0 \\ \sqrt{2E}\,\xi_{\uparrow} \end{pmatrix} = \begin{pmatrix} u_- \\ u_+ \end{pmatrix} = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

$$u_{\downarrow} = \begin{pmatrix} \sqrt{E+p}\,\xi_{\downarrow} \\ \sqrt{E-p}\,\xi_{\downarrow} \end{pmatrix} = \begin{pmatrix} \sqrt{2E}\,\xi_{\downarrow} \\ 0 \end{pmatrix} = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

$$\frac{1}{2}(1 \pm \gamma_5)$$

Spin projection on particle mom.

$\longrightarrow$ helicity

$$\hat{h} = \frac{i}{2} \varepsilon_{ijk} \hat{p}^i S^{jk} = \frac{1}{2} \begin{pmatrix} \hat{p}\vec{\sigma} & 0 \\ 0 & \hat{p}\vec{\sigma} \end{pmatrix}$$

$$\hat{p}^2 = 1$$

$u_\uparrow = u_R \qquad h = +1/2 \qquad$ Ex (?)

$u_\downarrow = u_L \qquad h = -1/2 \qquad \hat{p} = (\theta, \varphi)$

Inner and outer products

$$\varepsilon_{h'}^\dagger \varepsilon_h = 1 \qquad \eta_{h'}^\dagger \eta_h = 1$$

$$\sum_h \varepsilon_h \varepsilon_h^\dagger = \mathbb{1}_{2\times 2} \qquad \sum_h \eta_h \eta_h^\dagger = \mathbb{1}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\ 1)$$

inner

$$\bar{u}_{h'} u_h = 2m\, \delta_{h'h}$$

$$u_{h'}^\dagger u_h = 2E\, \delta_{h'h}$$

Ex (?)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

outer

$$\sum_h u_h \bar{u}_h = \sum_h \begin{pmatrix} \sqrt{p\sigma}\, \varepsilon_h \\ \sqrt{p\bar{\sigma}}\, \varepsilon_h \end{pmatrix} \left(\varepsilon_h^\dagger \sqrt{p\bar{\sigma}} \quad \varepsilon_h^\dagger \sqrt{p\sigma} \right)$$

$$\underbrace{\sum_h \varepsilon_h \varepsilon_h^\dagger = \mathbb{1}} \qquad = \begin{pmatrix} \sqrt{(p\sigma)(p\bar{\sigma})} & p\sigma \\ p\bar{\sigma} & \sqrt{(p\bar{\sigma})(p\sigma)} \end{pmatrix}$$

$$= \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} = \not{p} + m$$

$$\sum_h u_h \bar{u}_h = (\not{p} + m)$$

Similarly

$$\bar{v}_{h'} v_h = -2m\,\delta_{h'h}$$

$$v^\dagger_{h'} v_h = +2E\,\delta_{h'h}$$

$$\sum_h v_h \bar{v}_h = \not{p} - m$$

$$\bar{u}_{h'}(\vec{p})\, v_h(\vec{p}) = 0$$

$$\bar{u}_{h'}(\vec{p})\,\gamma^0\, v_h(-\vec{p}) = 0$$

Ex (?)

Quantization of Dirac field

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

Conjugate mom. $\pi = \dfrac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger$

Fermions: interchanging any 2 fermion fields gives a "−"

commutators $\longrightarrow$ anticommutators

$$\begin{cases} [\varphi(x), \varphi(y)] = [\pi(x), \pi(y)] = 0 \\ [\varphi(x), \pi(y)] = \delta^3(\vec{x} - \vec{y}) \end{cases}$$

$$\downarrow$$

$$\{\psi(x), \psi(y)\} = \{\pi(x), \pi(y)\} = 0$$

$$\underline{\{\psi_\alpha(x), \psi^\dagger_\beta(y)\}} = \delta_{\alpha\beta}\,\delta^3(\vec{x} - \vec{y})$$

$$\psi(\vec{x}) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big[b^s_{\vec{p}}\, u_s(\vec{p})\, e^{i\vec{p}\vec{x}} + c^{s+}_{\vec{p}}\, v_s(\vec{p})\, e^{-i\vec{p}\vec{x}} \Big]$$

$$\psi^+(\vec{y}) = \sum_{s'} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \Big[b^{s'+}_{\vec{q}}\, u^+_{s'}(\vec{p})\, e^{-i\vec{p}\vec{y}} + c^{s'}_{\vec{q}}\, v^+_{s'}(\vec{q})\, e^{+i\vec{q}\vec{y}} \Big]$$

$$\boxed{\{b^s_{\vec{p}}, b^{s'+}_{\vec{q}}\} = \{c^s_{\vec{p}}, c^{s'+}_{\vec{q}}\} = \underline{\delta_{ss'}}\,\underline{(2\pi)^3}\,\underline{\delta^3(\vec{p}-\vec{q})}}$$

all other $\{\,,\}=0$

$$\Downarrow$$

$$\{\psi(\vec{x}), \psi^+(\vec{y})\} = \sum_{s,s'} \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} \frac{1}{\sqrt{2E_p\, 2E_q}}$$

$$\times \Big\{ \{b^s_{\vec{p}}, b^{s'+}_{\vec{q}}\}\, u_s(\vec{p})\, u^+_{s'}(\vec{q})\, e^{i(\vec{p}\vec{x}-\vec{q}\vec{y})} + \{c^{s+}_{\vec{p}}, c^{s'}_{\vec{q}}\}\, v_s(\vec{p})\, v^+_{s'}(\vec{q})\, e^{-i(\vec{p}\vec{x}-\vec{q}\vec{y})} \Big\}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} \Big\{ (\not{p}+m)\gamma^0 e^{i\vec{p}(\vec{x}-\vec{y})} + (\not{p}-m)\gamma^0 e^{-i\vec{p}(\vec{x}-\vec{y})} \Big\}$$

$$\underline{= \delta^3(\vec{x}-\vec{y})}$$

OK

$$H = \pi \dot{\psi} - \mathcal{L} = \bar{\psi} \underbrace{\left(-i\gamma^i \partial_i + m\right) \psi}$$

$\downarrow$

T^{00}

$$\left\{ \begin{aligned} &\partial_i = \frac{\partial}{\partial x^i} \\ &\partial_i e^{i\vec{p}\vec{x}} = \partial_i e^{-ip_i x^i} = \frac{\partial}{\partial x^i} e^{-ip_i x^i} \\ &\qquad = -ip_i e^{i\vec{p}\vec{x}} \\ &\partial_i e^{-ip^i x_i} = +ip^i e^{i\vec{p}\vec{x}} \end{aligned} \right.$$

$$g^i{}_i = -1$$

$$\left(-i\gamma^i \partial_i + m\right) \psi = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}$$

$$\times \left\{ b_p^s \underbrace{\left(-i\gamma^i \partial_i + m\right) u_s(\vec{p}) e^{i\vec{p}\vec{x}}}_{=} \right.$$

$$\underbrace{\left(-\gamma^i p_i + m\right)}_{= \gamma^0 E_p} \quad — // — \quad (\not{p} - m) u = 0$$

$$= \left(\gamma^0 E_p + \gamma^i p_i - m\right) u$$

$$\left. + c_p^{s\dagger} \underbrace{\left(-i\gamma^i \partial_i + m\right) v_s(\vec{p}) e^{-i\vec{p}\vec{x}}}_{= -\gamma^0 E_p v e^{-i\vec{p}\vec{x}}} \right\}$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \gamma^0 \left[b_p^s u_s(\vec{p}) e^{i\vec{p}\vec{x}} - c_p^{s\dagger} v_s(\vec{p}) e^{-i\vec{p}\vec{x}} \right]$$

$$H = \int d^3x \, \psi^\dagger \gamma^0 \left(-i\gamma^i \partial_i + m\right) \psi$$

$$= \sum_{s,s'} \int d^3x \int \frac{d^3p \, d^3q}{(2\pi)^6} \sqrt{\frac{E_p}{4E_q}} \left[b_q^{s'\dagger} u_{s'}^\dagger(q) e^{-i\vec{q}\vec{x}} + c_q^{s'} v^\dagger(q) e^{i\vec{q}\vec{x}} \right]$$

$$\cdot\left[b_p^s u_s(\vec p)\, e^{i\vec p \vec x} - c_p^{s+} v_s(\vec p)\, e^{-i\vec p\vec x}\right]$$

$$1).\quad \int d^3\vec x\, e^{i\vec x(\vec p-\vec q)} = (2\pi)^3\delta^3(\vec p-\vec q)$$

$$= \sum_{s,s'}\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\Big\{ b_{\vec p}^{s'+} b_{\vec p}^{s} \underbrace{u_{s'}^{+}(\vec p)\,u_s(\vec p)}_{= 2E_p\delta_{ss'}}$$

$$- c_{\vec p}^{s'} c_{\vec p}^{s+} \underbrace{v_{s'}^{+}(\vec p)\,v_s(\vec p)}_{2E_p\delta_{s's}}$$

$$- b_{-\vec p}^{s'+} c_{\vec p}^{s+} \underbrace{u_{s'}^{+}(-\vec p)\,v_s(\vec p)}_{=0}$$

$$+ c_{-\vec p}^{s'} b_{\vec p}^{s} \underbrace{v_{s'}^{+}(-\vec p)\,u_s(\vec p)}_{=0}\Big\}$$

$$= \int\frac{d^3p}{(2\pi)^3} E_p\left[b_p^{s+}b_p^s - c_p^s c_p^{s+}\right]$$

$$= \int\frac{d^3\vec p}{(2\pi)^3} E_p\left[\underline{b_p^{s+}b_p^s + c_p^{s+}c_p^s} \underbrace{-(2\pi)^3\delta^3(0)}_{\text{neg. } \infty \text{ vacuum } E}\right]$$

bosons: positive ∞

We diagonalized the H

Define $|0\rangle$

Define 1-particle states