Lecture 16

What we did in the last lecture

→ Starting from Dirac's idea

$$\text{"}\sqrt{(\Box + m^2)\phi = 0}\text{"} \longrightarrow (i\partial_\mu \gamma^\mu - m)\underline{\underline{\psi}} = 0$$

$$\Box = \partial_\mu \partial^\mu$$

Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \gamma^{0\,2} = \underline{\mathbb{1}} \quad \gamma^{1\,2} = -\mathbb{1}$$

Then we found group generators

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$$

Obey Lie algebra

$$[S^{\mu\nu}, S^{\alpha\beta}] = g^{\nu\alpha}S^{\mu\beta} - g^{\mu\alpha}S^{\nu\beta} + g^{\mu\beta}S^{\nu\alpha} - g^{\nu\beta}S^{\mu\alpha}$$

Introduced Dirac spinors; transform as

$$\psi^\alpha(x) \longrightarrow S[\Lambda]^\alpha{}_\beta \, \psi^\beta(\Lambda^{-1}x)$$

$$\Lambda = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}\underline{\underline{M^{\rho\sigma}}}\right]$$

$$S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}\underline{\underline{S^{\rho\sigma}}}\right]$$

Ω defines the transf.

We found a peculiar property:

Under 2π - rotation $\psi \longrightarrow -\psi$

To construct invariant Lagrangian and action

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

$$(S^{\mu\nu})^\dagger = -\gamma^0 S^{\mu\nu} \gamma^0$$

$$S[\Lambda]^{+} = \gamma^0 S[\Lambda]^{-1}\gamma^0$$

Dirac adjoint $\bar{\psi} = \psi^{+}\gamma^0$

We could show that $\bar{\psi}\psi$ – scalar

$$\bar{\psi}\psi = \psi^{+}\gamma^0\psi \longrightarrow \psi^{+}\underbrace{S[\Lambda]^{+}\gamma^0 S[\Lambda]}_{\gamma^0 S[\Lambda]^{-1}\gamma^0\gamma^0 S[\Lambda] = \gamma^0}\psi$$

$$= \bar{\psi}\psi(\Lambda^{-1}x)$$

$\bar{\psi}\gamma^{\mu}\psi$ — Lorentz vector

$\bar{\psi}S^{\mu\nu}\psi$ — tensor

$\hookrightarrow$ $\mathcal{L} = \bar{\psi}(i\partial_{\mu}\gamma^{\mu} - m)\psi$ is Lorentz inv.

$S = \int d^4x\, \mathcal{L}(x)$ —//—

We introduced explicit repres. of Clifford alg

Weyl (Chiral) $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$1 = 1_{2\times 2}$ $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Dirac $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

Related by unitary transformation

$$\gamma^{\mu}_{W} = U\gamma^{\mu}_{D}U^{+}, \quad \gamma^{\mu}_{D} = U^{+}\gamma^{\mu}_{W}U$$

$$U^{+}U = 1 \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad U^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Exercise to follow through

Majorana basis

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

Let's work out what Weyl spinors are

— " — Majorana spinors

Weyl: $S[\Lambda_{rot}] = \begin{pmatrix} e^{i\vec{\varphi}\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\varphi}\vec{\sigma}/2} \end{pmatrix}$

$S[\Lambda_{boost}] = \begin{pmatrix} e^{\vec{\chi}\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\vec{\sigma}/2} \end{pmatrix}$

$\vec{\chi}, \vec{\varphi}$ real vectors

$\psi \to$ complex object

$S[\Lambda]$ is diagonal

If we split $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \longrightarrow \begin{pmatrix} u_- \\ u_+ \end{pmatrix}$

u_- and u_+ transform separately

$u_\pm \xrightarrow{rot} e^{\frac{i\vec{\varphi}\vec{\sigma}}{2}} u_\pm \qquad u_\pm \xrightarrow{boost} e^{\pm\vec{\chi}\vec{\sigma}/2} u_\pm$

Rewrite Dirac eq. in u_+ and u_-

$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \qquad$ Slash: $\not{\partial} = \partial_\mu \gamma^\mu$

$\not{a} = a_\mu \gamma^\mu$

$\mathcal{L} = i u_-^\dagger \bar{\sigma}^\mu \partial_\mu u_- + i u_+^\dagger \sigma^\mu \partial_\mu u_+$

$$- m\left(u_+^\dagger u_- + u_-^\dagger u_+\right)$$

$$\sigma^\mu = (1, \sigma^i) \qquad \bar{\sigma}^\mu = (1, -\sigma^i)$$

The mass mixes u_+ and u_-

For a massless fermion u_- and u_+ completely decouple

Weyl eqs
$$\begin{cases} i\bar{\sigma}^\mu \partial_\mu u_- = 0 \\ i\sigma^\mu \partial_\mu u_+ = 0 \end{cases}$$

There is a representation-independent method of defining the chiral spinors

Define $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\begin{cases} \{\gamma_5, \gamma^\mu\} = 0 \\ \gamma_5^2 = 1 \end{cases}$$

Exercise

Weyl : $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

γ_5 invariant under Lorentz transf.

$$[S^{\mu\nu}, \gamma_5] = 0$$

Ex

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

$$P_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\overset{P_L}{P_-}\psi = \begin{pmatrix} u_- \\ 0 \end{pmatrix} = \psi_- = \begin{pmatrix} u_L \\ 0 \end{pmatrix} = \psi_L$$

$$\overset{P_R}{P_+}\psi = \begin{pmatrix} 0 \\ u_+ \end{pmatrix} = \psi_+ = \begin{pmatrix} 0 \\ u_R \end{pmatrix} = \psi_R$$

$$P_L^2 = P_L \qquad P_R^2 = P_R \qquad P_R P_L = P_L P_R = 0$$

Ex

$\Psi_L \to$ left-handed
$\Psi_R \to$ right-handed

Discrete symmetries

Until now $\Lambda^{\mu\nu} = g^{\mu\nu} + \omega^{\mu\nu}$ $\quad \omega \to 0$

There exist two discrete trans.

Parity $\mathbb{P}$: $x^0 \to x^0$; $\vec{x} \to -\vec{x}$ $\quad \mathbb{P}^2 = \mathbb{1}$

Time reversal T: $x^0 \to -x^0$; $\vec{x} \to \vec{x}$ $\quad T^2 = \mathbb{1}$

Here we consider $\mathbb{P}$

*) rotations $\vec{x} \times \vec{y} \to (-\vec{x}) \times (-\vec{y}) = \vec{x} \times \vec{y}$
do not flip sign under $\mathbb{P}$

**) boosts change sign under $\mathbb{P}$

$$u_\pm \xrightarrow{\text{rot}} e^{\frac{i\vec{\varphi}\vec{\sigma}}{2}} u_\pm$$

$$u_\pm \xrightarrow{\text{boost}} e^{\pm\frac{\vec{\chi}\vec{\sigma}}{2}} u_\pm$$

$$u_\pm \xrightarrow{\mathbb{P}} u_\mp$$

$$\Psi_\pm(\vec{x}, t) \xrightarrow{\mathbb{P}} \Psi_\mp(-\vec{x}, t)$$

$$\Psi = \begin{pmatrix} u_- \\ u_+ \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbb{P} : \Psi(\vec{x}, t) \longrightarrow \gamma^0 \Psi(-\vec{x}, t)$$

What does P do on Dirac eq.?

$$(i\not{\partial}-m)\psi=0=(i\partial^0\gamma^0-i\vec{\partial}\vec{\gamma}-m)\psi=0$$

$$\xmapsto{P}(i\partial^0\gamma^0+i\vec{\partial}\vec{\gamma}-m)\gamma^0\psi$$

$$=\gamma^0(i\not{\partial}-m)\psi=0$$

How do bilinears behave under P?

$$\bar{\psi}\psi=\psi^\dagger\gamma^0\psi\longmapsto(\psi^\dagger\gamma^0)\gamma^0\gamma^0\psi$$
$$=\bar{\psi}\psi(-\vec{x},t)$$

$\bar{\psi}\psi$ — scalar

$P:\bar{\psi}\gamma^\mu\psi$ $\qquad \bar{\psi}\gamma^0\psi\to\bar{\psi}\gamma^0\psi(-\vec{x},t)$

$$\bar{\psi}\vec{\gamma}\psi\to\psi^\dagger\gamma^0\gamma^0\vec{\gamma}\gamma^0\psi=-\bar{\psi}\vec{\gamma}\psi$$

$\bar{\psi}\gamma^\mu\psi$ — vector

$P:\bar{\psi}S^{\mu\nu}\psi\to$ tensor

How we can also build bilinears w. γ_5

$$\bar{\psi}\gamma_5\psi \qquad \bar{\psi}\gamma^\mu\gamma_5\psi$$

$$P:\bar{\psi}\gamma_5\psi\to(\psi^\dagger\gamma^0)\gamma^0\gamma_5\gamma^0\psi=-\bar{\psi}\gamma_5\psi(-\vec{x},t)$$

$\bar{\psi}\gamma_5\psi$ — pseudoscalar

Example: $\underbrace{\vec{E}\cdot\vec{B}}$ $\qquad \vec{E}$ true vector

$\qquad\qquad\qquad\qquad \vec{B}$ axial vector

$P: \bar{\psi}\gamma^\mu\gamma_5\psi \quad \bar{\psi}\gamma^0\gamma_5\psi \xrightarrow{P} -\bar{\psi}\gamma^0\gamma_5\psi(-\vec{x},t)$

$\bar{\psi}\vec{\gamma}\gamma_5\psi \xrightarrow{P} +\bar{\psi}\vec{\gamma}\gamma_5\psi(-\vec{x},t)$

$\bar{\psi}\gamma^\mu\gamma_5\psi$ - axial vector

$P_\pm \rightarrow P_\mp \qquad \psi_L \xrightarrow{P} \psi_R$

$P_{L,R} \rightarrow P_{R,L}$

We have seen that Dirac eq. for massive fermions mixes chiralities u_+ and u_-

u_R and u_L

Theories that make no distinction between u_L and u_R called <u>vector-like</u> (spinor) QED

Theories that treat u_L and u_R differently $\rightarrow$ called <u>chiral</u>

Weak sector of Standard Model

Weak charged interaction (β-decay) operates only upon u_L

not on u_R

Majorana fermions

Majorana basis

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

all γ^μ's are purely imaginary

$S^{\mu\nu}$ is purely real

If you start with a real spinor ψ
it stays that way $\psi^* = \psi$
↳ Majorana spinors

Charge conjugation

$C^\dagger C = 1$ $\qquad$ $C^\dagger \gamma^\mu C = -(\gamma^\mu)^*$

$\psi^c = C\psi^*$
↑
charge-conjugate

Majorana basis
$C = 1$
Weyl $C = i\gamma^2$

Under Lorentz transf.

$$\psi^c \longrightarrow C\, S[\Lambda]^* \psi^* = S[\Lambda]\psi^c$$

$$C \underbrace{\gamma^{\mu *}\gamma^{\nu *}}_{S^*} = -\gamma^\mu C \gamma^{\nu *} = +\underbrace{\gamma^\mu\gamma^\nu}_{S} C$$

$\gamma^\mu C = -C\gamma^{\mu *}$

$$C: (i\partial\!\!\!/ - m)\psi = 0 \Rightarrow (-i\partial\!\!\!/^* - m)\psi^* = 0$$

$$\Rightarrow C(-i\partial\!\!\!/^* - m)\psi^* = 0 \Rightarrow (i\partial\!\!\!/ - m)\psi^c = 0$$

Majorana condition: $\psi^c = \psi$

This means that we can operate with

a purely real spinor ψ

Upon quantization it leads to a particle that is its own antiparticle
(Complex field $\rightarrow$ it is not)

$\hookrightarrow$ this theory cannot couple to photons

Majorana fermions can only be neutral

M.f. have not been observed in nature (yet)

$\rightarrow$ active searches!

neutrino masses

neutrino magnetic moments

neutrinoless double beta decay

$\rightarrow$ exotic searches

$n - \bar{n}$ oscillations

What Majorana basis is a solution to?

Recall $\mathcal{L} = u_L^\dagger i \sigma^\mu \partial_\mu u_L + u_R^\dagger i \bar{\sigma}^\mu \partial_\mu u_R$

$$- m \left(u_L^\dagger u_R + u_R^\dagger u_L \right)$$

Majorana mass term allows to avoid this mixing

$$\bar{\psi}_L = \psi_L^\dagger \gamma_0 = \psi^\dagger \frac{1-\gamma_5}{2} \gamma_0$$

$$= \bar{\psi} P_R$$

$$m \overline{\psi_L} \psi_L = m \bar{\psi} P_R P_L \psi = 0$$

$C \psi_L$ $\qquad$ $\underline{C^+ \gamma_5^T C = \gamma_5}$

Majorana mass term $-\frac{1}{2} m \underbrace{(\psi_L^c)^T \psi_L}$

$$\psi = \begin{pmatrix} \vdots \end{pmatrix} \qquad \psi^T = (\cdots)$$

<u>Important in SM</u>

only left-handed neutrinos are present (electrons out of muon decay are always left)

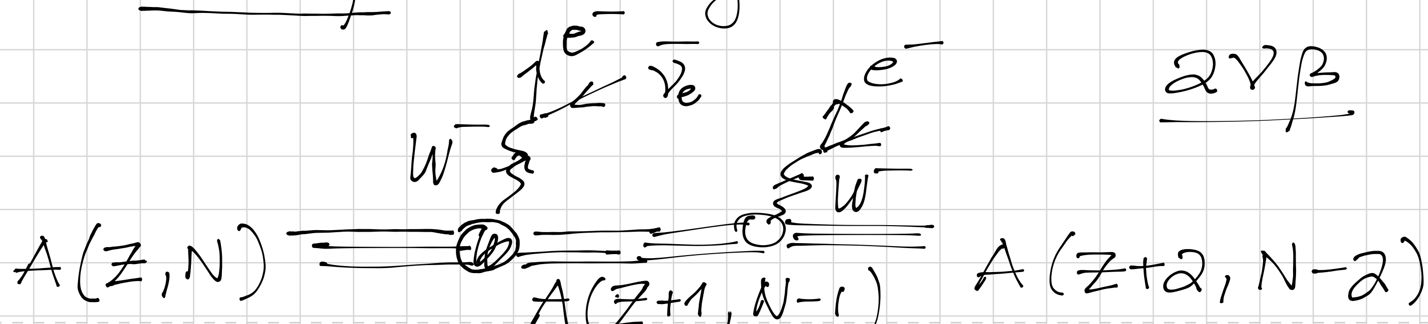
We know ν's have masses (ν oscillations)

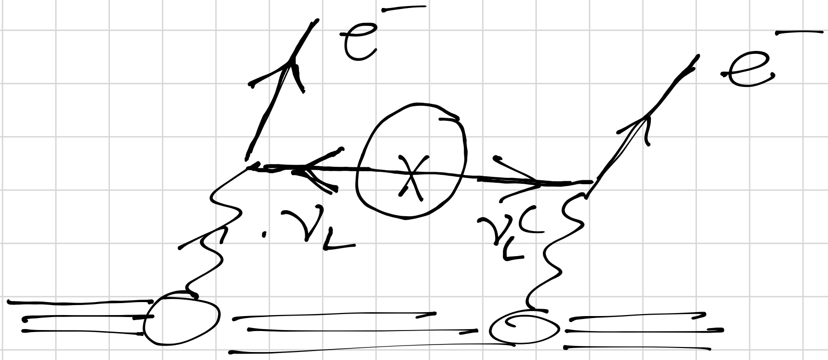
$-m \bar{\nu}_R \nu_L$ is not possible in SM because no ν_R

$$-\frac{1}{2} (\nu_L^c)^T \nu_L m$$

<u>$0\nu\beta$ decay</u>

<u>$2\nu\beta$</u>





Majorana m.t.

$\longleftrightarrow\!\!\!\!\!\!\times$

$\nu_L = \bar{\nu}_L^c$

Majorana m.t. violates lepton number conservation by 2 units

$0\nu\beta\beta$ process strictly forbidden if ν's are not Majorana particles!