Lecture 15

What we did so far

Lorentz inv. $\quad x^\mu \to x'^\mu = \underbrace{\Lambda^{\mu\nu}}_{4\times 4 \text{ matrix}} x_\nu$

$$\underline{\Lambda^T g \Lambda = g} \qquad \Lambda^{\mu\nu} g_{\mu\alpha} \Lambda^{\alpha\beta} = g^{\nu\beta}$$

Spin 0 fields $\quad \phi \to \phi' = \phi(\Lambda^{-1}x)$

$\downarrow$ We quantized $\phi \to$ infinite # of harmonic oscillators

$\hat{a}_k, \hat{a}_k^+ \longrightarrow$ spin-0 particles

From here $\to$ extended to vector field

$$\Lambda : \quad A^\mu(x) \longrightarrow \Lambda^{\mu\nu} A_\nu(\Lambda^{-1}x)$$

To quantize $\to$ introduced polarization v.

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left[\hat{a}_k \, \varepsilon^\mu_\lambda(\underline{k})\, e^{-ikx} + \hat{a}_k^+ \, \varepsilon^{\mu *}_\lambda(k)\, e^{ikx} \right] = A^\mu(x)$$

Complication: a 4-vector has 4 components
massive vector $\longrightarrow$ only 3 !
One condition to remove the redundant spin-0 component

$$m^2 (\partial_\mu A^\mu) = 0$$

Massless case didn't work

↳ Gauge invariance → only 2 physical polarizations

Build interacting field theory
scalars + photons
(complex)

$$A^\mu \to A^\mu + \partial^\mu \alpha$$

$$\underline{\partial^\mu} \to \underline{D^\mu} = \partial^\mu + ieA^\mu$$

Scalar QED

Spin-½ (Dirac theory)

Non-zero spin fields transform non-trivially under Lorentz group

$$\varphi^\alpha(x) \longrightarrow D[\Lambda]^\alpha{}_\beta \, \varphi^\beta(\Lambda^{-1}x)$$

$D[\Lambda]$ form a representation of Lorentz group

$$D[\Lambda_1]\,D[\Lambda_2] = D[\Lambda_1\Lambda_2]$$

↳ $D[\Lambda^{-1}] = D[\Lambda]^{-1}$

$$D[1] = 1$$

How do we find different representations?

We consider infinitesimal transformations
→ study the resulting Lie algebra.

$$\Lambda^{\mu\nu} = g^{\mu\nu} + \omega^{\mu\nu}$$

$$g^{\mu\nu} = \Lambda^{\mu\alpha}\Lambda^{\nu\beta} g_{\alpha\beta} = (g^{\mu\alpha} + \omega^{\mu\alpha})(g^{\nu}{}_{\alpha} + \omega^{\nu}{}_{\alpha})$$

$$= g^{\mu\nu} + \omega^{\mu\nu} + \omega^{\nu\mu} + \underbrace{\omega^{\mu\alpha}\omega^{\nu}{}_{\alpha}}_{\to 0}$$

$$\to \quad \omega^{\mu\nu} + \omega^{\nu\mu} = 0$$

ω – antisymmetric

Antisymm. 4×4 matrix → 6 indep. comp.

6 LT : 3 rotations + 3 boosts

6 indep. 4×4 matrices that form a basis

$M^{\rho\sigma}$ antisym. $\overbrace{\text{matrix}}^{4\times4}$ of 4×4 matrices

$$M^{\rho\sigma} = \begin{pmatrix} 0_{4\times4} & M^{01} & M^{02} & M^{03} \\ -M^{01} & 0 & M^{12} & M^{13} \\ & & 0 & M^{23} \\ & -//- & & 0 \end{pmatrix}$$

$$(M^{\underbrace{\rho\sigma}})^{\underbrace{\mu\nu}} = g^{\rho\mu} g^{\sigma\nu} - g^{\rho\nu} g^{\sigma\mu}$$

counts elements

elements of 4×4 matrices

$$(M^{01})^{\mu\nu} = g^{0\mu} g^{1\nu} - g^{0\nu} g^{1\mu}$$

$$\left.\begin{aligned} g^{00} &= +1 \\ g^{11} &= -1 \end{aligned}\right| \quad 0 \text{ otherw.}$$

$$\mathcal{M}^{01} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ boost in x-direction

$$(\mathcal{M}^{12})^{\mu\nu} = g^{1\mu} g^{2\nu} - g^{1\nu} g^{2\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$g^{11} g^{22}$

rotation in (x,y) plane

What is the use of this representation?

Express <u>any</u> $\omega^{\mu\nu}$ through $\mathcal{M}^{\rho\sigma}$

$$\omega^{\mu\nu} = \frac{1}{2} \underbrace{\Omega_{\rho\sigma}}_{\text{6 numbers}} \cdot \underbrace{(\mathcal{M}^{\rho\sigma})^{\mu\nu}}_{\text{representation (basis)}}$$

(3 Euler angles
3 boosts)

The six $\mathcal{M}^{\rho\sigma}$ are called generators of the Lorentz group

They obey Lie algebra:

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = g^{\sigma\tau}\mathcal{M}^{\rho\nu} - g^{\rho\tau}\mathcal{M}^{\sigma\nu} + g^{\rho\nu}\mathcal{M}^{\sigma\tau} - g^{\sigma\nu}\mathcal{M}^{\rho\tau}$$

$\downarrow$

Any Lorentz transf. is expressed as

$$\Lambda = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right]$$

We want to find other representations that Lie algebra

Dirac eq. $= \sqrt{\text{KG eq.}}$

$$(\Box + m^2)\varphi = 0 \longrightarrow (i\partial_\mu \gamma^\mu - m)\psi = 0$$

$$\times(-i\partial_\mu \gamma^\mu - m) \longrightarrow \left(\partial_\mu \partial_\nu \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \cancel{\frac{1}{2}[\gamma^\mu, \gamma^\nu]}\right) + m^2\right)\psi = 0$$

Recover KG eq. if:

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \qquad \mu \neq \nu$$
$$(\gamma^0)^2 = \mathbb{1}, \quad (\gamma^i)^2 = -\mathbb{1}, \quad i = 1,2,3$$

Simplest representation $\longrightarrow$ 4×4 matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Pauli matrices)

Weyl (chiral) representation

Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i \text{ —— // ——}$$

Different representations related by a unitary transf.

$$\gamma_W = U \gamma_D U^+ \qquad\qquad U^+ U = \mathbb{1}$$

$$\gamma_D = U^+ \gamma_D U$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \longrightarrow \quad \underline{\text{Exercise}}$$

Out of matrices γ^μ can define a new antisymm. matrix of matrices

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0, & \rho = \sigma \\ \underline{\frac{1}{2}\gamma^\rho\gamma^\sigma}, & \rho \neq \sigma \end{cases}$$

$$\gamma^\rho\gamma^\sigma = 2g^{\rho\sigma} - \gamma^\sigma\gamma^\rho$$

Properties:

$$*) \quad [S^{\mu\nu}, \gamma^\lambda] = \gamma^\mu g^{\nu\lambda} - \gamma^\nu g^{\mu\lambda}$$

<u>Exercise</u> the trick: $\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$

$$**) \quad [S^{\mu\nu}, S^{\alpha\beta}] = g^{\nu\alpha} S^{\mu\beta} - g^{\mu\alpha} S^{\nu\beta} + g^{\mu\beta} S^{\nu\alpha} - g^{\nu\beta} S^{\mu\alpha}$$

<u>Exercise</u>

Trick: $[S^{\mu\nu}, \frac{1}{2}\gamma^\alpha\gamma^\beta] \longleftrightarrow [S^{\mu\nu}, \gamma^\alpha]$

$\longleftrightarrow [S^{\mu\nu}, \gamma^\beta]$

$S^{\mu\nu}$ are 4×4 matrices

We need a field $(S^{\mu\nu})^{\alpha\beta}$ act upon

$\longrightarrow$ Dirac spinors

$$\psi^{\alpha}(x) \xrightarrow{\Lambda} S[\Lambda]^{\alpha}{}_{\beta}\, \psi^{\beta}(\Lambda^{-1}x)$$

$\alpha = 1,2,3,4$

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right)$$

different

$$S[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma} \underline{S^{\rho\sigma}}\right)$$

same $\underline{\Lambda}$

<u>Consider rotation</u>

$$S^{ij} = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0\end{pmatrix} = -\frac{i}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$$

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$$

$$\Omega_{ij} = -\varepsilon_{ijk}\varphi^k \qquad \vec{\varphi}$$

$$\rightarrow S[\Lambda] = \begin{pmatrix} \exp\left(i\frac{\vec{\varphi}\vec{\sigma}}{2}\right) & 0 \\ 0 & \exp\left(i\frac{\vec{\varphi}\vec{\sigma}}{2}\right)\end{pmatrix}$$

$$\exp\left(i\vec{\theta}\vec{\sigma}\right) = \sum_{k=0}^{\infty}\frac{(i\vec{\theta}\vec{\sigma})^k}{k!}$$

$$= \sum_{k=0}^{\infty}\frac{(i\vec{\theta}\vec{\sigma})^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{(i\vec{\theta}\vec{\sigma})^{2k+1}}{(2k+1)!}$$

$\Big\{\; i^{2k} = (-1)^k;\; (\vec{\theta}\vec{\sigma})^{2k} = \left((\vec{\theta}\cdot\vec{\sigma})^2\right)^k = \theta^{2k}$

$$\theta^i\sigma^i\cdot\theta^j\sigma^j = \theta^i\theta^j(\delta^{ij} + \cancel{i\varepsilon^{ijk}\sigma^k}) = \vec{\theta}^2$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!}\mathbb{1} + i\sum_{k=0}^{\infty}\frac{(-1)^{2k}\theta^{2k+1}}{(2k+1)!}(\hat{e}_\theta\cdot\vec{\sigma})$$

$$= \cos\theta\cdot\mathbb{1} + i\sin\theta\,(\hat{e}_\theta\cdot\vec{\sigma})$$

$\vec{\varphi} = (0,0,2\pi)$ (rotation by 2π around z)

$$S[\Lambda] = \begin{pmatrix}\exp(i\pi\sigma^3) & 0\\ 0 & \exp(i\pi\sigma^3)\end{pmatrix} = -\mathbb{1}_{4\times4}$$

2π rotation $\psi^\alpha \longrightarrow -\psi^\alpha$

<u>Boosts</u> $S^{0i} = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$

$$\Omega^{0i} = \chi^i$$

$$S[\Lambda] = \begin{pmatrix}\exp[\frac{\vec{\chi}\vec{\sigma}}{2}] & 0\\ 0 & \exp[-\frac{\vec{\chi}\vec{\sigma}}{2}]\end{pmatrix}$$

<u>Attn:</u> Rotation is unitary

$$S[\Lambda]^\dagger S[\Lambda] = \mathbb{1}$$

Boost is not unitary

$S^\dagger S = \mathbb{1}$ $\longrightarrow$ if $(S^{\rho\sigma})^\dagger = -S^{\rho\sigma}$

$$S = \exp[\tfrac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}]$$

But: $(S^{\rho\sigma})^{\dagger} = -\frac{1}{4}[(\gamma^{\rho})^{\dagger}, (\gamma^{\sigma})^{\dagger}]$

$$\left(\begin{array}{l} (\gamma^{0})^{\dagger} = \gamma^{0} \\ (\gamma^{i})^{\dagger} = -\gamma^{i} \end{array} \right)$$

No finite dimensional unitary representation of the Lorentz group

$\downarrow$

We want to construct action from fields ψ

Bilinears: ψ, $\psi^{\dagger} = (\psi^{*})^{T}$

$$\psi^{\alpha} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi^{\dagger} = (\psi_1^{*}\ \psi_2^{*}\ \psi_3^{*}\ \psi_4^{*})$$

Will $\psi^{\dagger}\psi$ do?

$$\psi \to S[\Lambda]\,\psi(\Lambda^{-1}x)$$

$$\psi^{\dagger} \to \psi^{\dagger}(\Lambda^{-1}x)\, S[\Lambda]^{\dagger}$$

$$\psi^{\dagger}\psi \to \psi^{\dagger}(\Lambda^{-1})\, \underbrace{S[\Lambda]^{\dagger} S[\Lambda]}_{\neq \mathbb{1}}\, \psi(\Lambda^{-1}x)$$

$\psi^{\dagger}\psi$ is <u>not</u> a Lorentz scalar

$$\left. \begin{array}{l} (\gamma^{0})^{\dagger} = \gamma^{0} \\ (\gamma^{i})^{\dagger} = -\gamma^{i} \end{array} \right] \longrightarrow (\gamma^{\mu})^{\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0}$$

$$[S^{\mu\nu}]^\dagger = \frac{1}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}]$$
$$= -\gamma^0 S^{\mu\nu}\gamma^0$$
$$\downarrow$$
$$S[\Lambda]^\dagger = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right) = \gamma^0 S[\Lambda]^{-1}\gamma^0$$
$$\downarrow$$
Define the Dirac adjoint
$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$$
$\rightarrow$ $\bar{\psi}\psi$ is a Lorentz scalar

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi \rightarrow \psi^\dagger(\Lambda^{-1}x)\underbrace{S[\Lambda]^\dagger}_{\gamma^0 S[\Lambda]^{-1}\gamma^0}\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x)$$
$$= \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x)$$

2 more exercises:

*) Show that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector

**) Show that $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ —//— a tensor

Reccomended read:

M. Schwartz, QFT and Standard Model, Chapter 10