Exercise sheet 9
Theoretical Physics 3 : QM WS2020/2021
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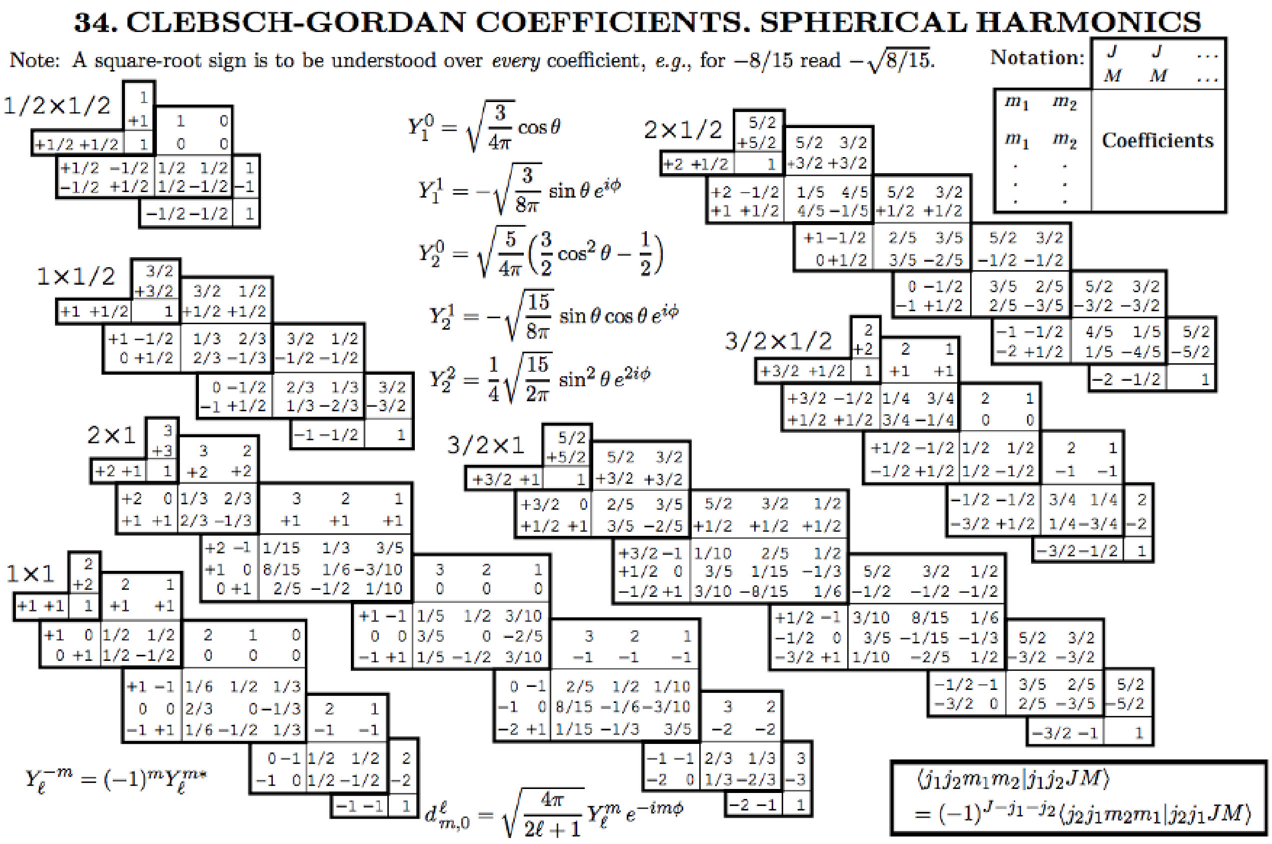
Exercise 1. (40 points)

In this exercise we will practice how to couple two angular momenta j_1 and j_2, using the Clebsch-Gordan Table.

Recall that the coupled states which are characterized by the total angular momentum J and its projection M can be expanded via the completeness relation in the uncoupled basis:

$$|JM\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1 j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | JM\rangle.$$

The expansion coefficients, $\langle j_1 m_1 j_2 m_2 | JM\rangle$, are the Clebsch-Gordan coefficients which can be found in the following table:


34. CLEBSCH-GORDAN COEFFICIENTS. SPHERICAL HARMONICS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation: J J ... / M M ... / m_1 m_2 / m_1 m_2 Coefficients

1/2×1/2: 1 / +1 | +1/2 +1/2: 1 | 1 0 / 0 0 | +1/2 −1/2: 1/2 1/2 | −1/2 +1/2: 1/2 −1/2 | 1 / −1 | −1/2 −1/2: 1

1×1/2: 3/2 / +3/2 | +1 +1/2: 1 | 3/2 1/2 / +1/2 +1/2 | +1 −1/2: 1/3 2/3 | 0 +1/2: 2/3 −1/3 | 3/2 1/2 / −1/2 −1/2 | 0 −1/2: 2/3 1/3 | −1 +1/2: 1/3 −2/3 | 3/2 / −3/2 | −1 −1/2: 1

2×1: 3 / +3 | +2 +1: 1 | 3 2 / +2 +2 | +2 0: 1/3 2/3 | +1 +1: 2/3 −1/3 | 3 2 1 / +1 +1 +1 | +2 −1: 1/15 1/3 3/5 | +1 0: 8/15 1/6 −3/10 | 0 +1: 2/5 −1/2 1/10 | 3 2 1 / 0 0 0 | +1 −1: 1/5 1/2 3/10 | 0 0: 3/5 0 −2/5 | −1 +1: 1/5 −1/2 3/10 | 3 2 1 / −1 −1 −1 | 0 −1: 2/5 1/2 1/10 | −1 0: 8/15 −1/6 −3/10 | −2 +1: 1/15 −1/3 3/5 | 3 2 / −2 −2 | −1 −1: 2/3 1/3 | −2 0: 1/3 −2/3 | 3 / −3 | −2 −1: 1

1×1: 2 / +2 | +1 +1: 1 | 2 1 / +1 +1 | +1 0: 1/2 1/2 | 0 +1: 1/2 −1/2 | 2 1 0 / 0 0 0 | +1 −1: 1/6 1/2 1/3 | 0 0: 2/3 0 −1/3 | −1 +1: 1/6 −1/2 1/3 | 2 1 / −1 −1 | 0 −1: 1/2 1/2 | −1 0: 1/2 −1/2 | 2 / −2 | −1 −1: 1

$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$

$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$

$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{i\phi}$

$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{2i\phi}$

3/2×1: 5/2 / +5/2 | +3/2 +1: 1 | 5/2 3/2 / +3/2 +3/2 | +3/2 0: 2/5 3/5 | +1/2 +1: 3/5 −2/5 | 5/2 3/2 1/2 / +1/2 +1/2 +1/2 | +3/2 −1: 1/10 2/5 1/2 | +1/2 0: 3/5 1/15 −1/3 | −1/2 +1: 3/10 −8/15 1/6 | 5/2 3/2 1/2 / −1/2 −1/2 −1/2 | +1/2 −1: 3/10 8/15 1/6 | −1/2 0: 3/5 −1/15 −1/3 | −3/2 +1: 1/10 −2/5 1/2 | 5/2 3/2 / −3/2 −3/2 | −1/2 −1: 3/5 2/5 | −3/2 0: 2/5 −3/5 | 5/2 / −5/2 | −3/2 −1: 1

2×1/2: 5/2 / +5/2 | +2 +1/2: 1 | 5/2 3/2 / +3/2 +3/2 | +2 −1/2: 1/5 4/5 | +1 +1/2: 4/5 −1/5 | 5/2 3/2 / +1/2 +1/2 | +1 −1/2: 2/5 3/5 | 0 +1/2: 3/5 −2/5 | 5/2 3/2 / −1/2 −1/2 | 0 −1/2: 3/5 2/5 | −1 +1/2: 2/5 −3/5 | 5/2 3/2 / −3/2 −3/2 | −1 −1/2: 4/5 1/5 | −2 +1/2: 1/5 −4/5 | 5/2 / −5/2 | −2 −1/2: 1

3/2×1/2: 2 / +2 | +3/2 +1/2: 1 | 2 1 / +1 +1 | +3/2 −1/2: 1/4 3/4 | +1/2 +1/2: 3/4 −1/4 | 2 1 / 0 0 | +1/2 −1/2: 1/2 1/2 | −1/2 +1/2: 1/2 −1/2 | 2 1 / −1 −1 | −1/2 −1/2: 3/4 1/4 | −3/2 +1/2: 1/4 −3/4 | 2 / −2 | −3/2 −1/2: 1

$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$

$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$

$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$

a) *(25 p.)* Write down all the possible states $|JM\rangle$ in the basis $|j_1 m_1\rangle\,|j_2 m_2\rangle$ for the compositions $\frac{1}{2} \otimes 1$ and $1 \otimes 1$ (the symbol $\otimes$ stands for the coupling of two angular momenta).

b) *(15 p.)* Check explicitly that the decompositions of the state $|\frac{5}{2}, +\frac{1}{2}\rangle$ in the basis $|\frac{1}{2}m_1\rangle\,|1m_2\rangle\,|1m_3\rangle$ obtained from $(\frac{1}{2}\otimes 1)\otimes 1$ and $\frac{1}{2}\otimes(1\otimes 1)$ are the same.

Exercise 2. (30 points)

Consider a general spin-1/2 state

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

which is normalized $|a|^2 + |b|^2 = 1$.

a) *(10 p.)* Show that there always exists a direction in space $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ such that χ is the eigenstate of the spin component along this direction $S_{\vec{n}} = \vec{n}\cdot\vec{S}$ with eigenvalue $\hbar/2$.

b) *(15 p.)* Write θ and ϕ in terms of a and b.

c) *(5 p.)* Would an analogous result hold for higher spin states?

Hint: Count the number of degrees of freedom.

Exercise 3. (30 points)

a) *(15 p.)* Derive the spin matrices S_x, S_y, S_z in the basis $|s, s_z\rangle$ for $s = 1$.

b) *(15 p.)* Find the eigenvalues and the normalized eigenvectors of S_x and S_y in that basis.

Hint: The general relation $S_\pm|s, s_z\rangle = \hbar\sqrt{s(s+1) - s_z(s_z \pm 1)}|s, s_z \pm 1\rangle$ can we useful.