Relativistic QFT (Theo 6a): Exercise Sheet 9
Total: 100 points

15/01/2021

1. Axial current in Dirac theory (10 points)

Consider the axial transformation of the spinor field ψ,

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}, \tag{1}$$

and obtain the divergence of the axial current $J_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$:

$$\partial_\mu J_A^\mu = 2im\bar{\psi}\gamma_5\psi. \tag{2}$$

2. Plane wave solutions of Dirac equation (30 points)

The positive and negative energy plane wave solutions of the Dirac equation can be written as

$$\psi^{(E>0)} = u(p)e^{-ipx}, \quad \psi^{(E<0)} = v(p)e^{ipx} \tag{3}$$

respectively.

(a) Pasting (3) into the Dirac equation, show that the spinors $u(p)$ and $v(p)$ can be expressed as

$$u(p) = \begin{pmatrix} \sqrt{p_\mu\sigma^\mu}\xi \\ \sqrt{p_\mu\bar{\sigma}^\mu}\xi \end{pmatrix}, \quad v(p) = \begin{pmatrix} \sqrt{p_\mu\sigma^\mu}\eta \\ -\sqrt{p_\mu\bar{\sigma}^\mu}\eta, \end{pmatrix} \tag{4}$$

where ξ and η are 2×2 spinors, $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$.

(b) Consider the helicity operator

$$\hat{h}(\vec{p}) = \frac{1}{2}\begin{pmatrix} \vec{n}_p\vec{\sigma} & 0 \\ 0 & \vec{n}_p\vec{\sigma} \end{pmatrix}, \tag{5}$$

where $\vec{n}_p = \vec{p}/|\vec{p}| = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$ is the unit vector that is directed along the 3-momentum $\vec{p}$ of the moving particle. Find the eigenstates and corresponding eigenvectors of the helicity operator, i.e. the expressions for $u(p)_R$ and $u(p)_L$ for arbitrary directed particle 3-momentum $\vec{p}$.

(c) Knowing that the ξ and η spinors with definite helicities h have the following properties

$$\xi^\dagger_{h'}\xi_h = \delta_{h'h}, \quad \eta^\dagger_{h'}\eta_h = \delta_{h'h}, \tag{6}$$

show that u and v spinors satisfy the relations

$$\bar{u}_{h'}u_h = 2m\delta_{h'h}, \quad u^\dagger_{h'}u_h = 2E\delta_{h'h}, \quad \bar{v}_{h'}v_h = -2m\delta_{h'h}, \quad v^\dagger_{h'}v_h = 2E\delta_{h'h}, \tag{7}$$

$$\bar{u}_{h'}v_h = 0, \quad u^\dagger_{h'}(\vec{p})v_h(-\vec{p}) = 0. \tag{8}$$

3. Spin operator in Dirac theory (30 points)

The spin tensor in Dirac theory is defined as

$$S^{\mu\rho\sigma} = -i\bar{\psi}\gamma^\mu S^{\rho\sigma}\psi, \quad S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma]. \tag{9}$$

(a) Show that the current which corresponds to the third component of the spin is conserved, i.e.

$$\partial_\mu S^{\mu 12} = 0, \tag{10}$$

and the correspondent charge is

$$S_3 = \int d^3\vec{x} S^{012}. \tag{11}$$

(b) Find the secondary quantized expression for S_3.

4. Secondary quantized Dirac field in the interaction picture (20 points)

Using the secondary quantized Dirac fields in the Schrödinger picture,

$$\psi(\vec{x}) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b^s_{\vec{p}} u_s(\vec{p}) e^{i\vec{p}\vec{x}} + c^{s\dagger}_{\vec{p}} v_s(\vec{p}) e^{-i\vec{p}\vec{x}} \right], \tag{12}$$

$$\psi^\dagger(\vec{y}) = \sum_{s'} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \left[b^{s'\dagger}_{\vec{q}} u^\dagger_{s'}(\vec{q}) e^{-i\vec{q}\vec{y}} + c^{s'}_{\vec{q}} v_{s'}(\vec{q}) e^{i\vec{q}\vec{y}} \right], \tag{13}$$

along with the Hamiltonian

$$H = \sum_r \int \frac{d^3\vec{t}}{(2\pi)^3} E_t \left[b^{r\dagger}_{\vec{t}} b^r_{\vec{t}} + c^{r\dagger}_{\vec{t}} c^r_{\vec{t}} \right], \tag{14}$$

and anticommutation relations on the creation and annihilation operators

$$\begin{aligned} &\{b^s_q, b^{s'\dagger}_p\} = \{c^s_q, c^{s'\dagger}_p\} = \delta^{ss'} (2\pi)^3 \delta^3(\vec{p} - \vec{q}), \\ &\{b, b\} = \{b^\dagger, b^\dagger\} = \{c, c\} = \{c^\dagger, c^\dagger\} = \{b, c\} = \{b^\dagger, c^\dagger\} = \{b, c^\dagger\} = \{b^\dagger, c\} = 0 \end{aligned} \tag{15}$$

prove the commutation relations

$$[H, b_p] = -E_p b_p, \quad [H, c_p] = -E_p c_p, \quad [H, b^\dagger_p] = E_p b^\dagger_p, \quad [H, c^\dagger_p] = E_p c^\dagger_p, \tag{16}$$

and use them to evaluate the t-dependence of the Dirac field $\psi(\vec{x})$ from the equation

$$\frac{\partial \psi}{\partial t} = i[H, \psi] \tag{17}$$

5. Feynman propagator in Dirac theory (10 points)

Show that the Feynman propagator of the Dirac theory,

$$S_F^{\alpha\beta}(x-y) = i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{(\not{p} + m)^{\alpha\beta}}{p^2 - m^2 + i\epsilon}, \tag{18}$$

is the Green's function of Dirac equation:

$$(i\not{\partial}_x - m) S_F(x-y) = i\delta^{(4)}(x-y). \tag{19}$$