Relativistic QFT (Theo 6a): Exercise Sheet 8
Total: 100 points

8/01/2021

1. Gamma matrices in different representations: Dirac, Weyl, Majorana (20 points)

The gamma matrices in Dirac representation are

$$\gamma_D^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma_D^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_D^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{1}$$

where σ^i are 2×2 Pauli matrices, $\mathbb{1}$ is 2×2 identity matrix.
The transition to another representation can be done by the following unitarity transformation

$$\gamma^\mu_{\text{new}} = U \gamma^\mu_D U^\dagger \tag{2}$$

(a) (10 pt) Show that the gamma matrices in Weyl representation,

$$\gamma_W^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_W^5 = -\gamma_D^0, \tag{3}$$

where $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$, can be obtained from the ones in Dirac representation applying the following unitary transformation

$$U_{D\to W} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \tag{4}$$

(b) (10 pt) Show that the gamma matrices in Majorana representation,

$$\gamma_M^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma_M^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma_M^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \tag{5}$$

$$\gamma_M^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \quad \gamma_M^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \tag{6}$$

can be obtained from the ones in Dirac representation applying the following unitary transformation

$$U_{D\to M} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \sigma^2 \\ \sigma^2 & -\mathbb{1} \end{pmatrix}. \tag{7}$$

Use the property of Pauli matrices: $\sigma^i \sigma^j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma^k$, where δ and ε are the Kronecker and Levi-Civita symbols respectively.

2. Commutation relations for the Lorentz generators (20 points)

Prove that the Lorentz generators $S^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ obey the following commutational relations:

(a) (5 pt) $[S^{\mu\nu}, \gamma^\lambda] = \gamma^\mu g^{\nu\lambda} - \gamma^\nu g^{\mu\lambda}$,
(b) (10 pt) $[S^{\mu\nu}, S^{\alpha\beta}] = g^{\nu\alpha} S^{\mu\beta} - g^{\mu\alpha} S^{\nu\beta} + g^{\mu\beta} S^{\nu\alpha} - g^{\nu\beta} S^{\mu\alpha}$
(c) (5 pt) $[S^{\mu\nu}, \gamma^5] = 0$

3. Lorentz invariant spinor structures (30 points)

Consider the spinor Lorentz transformation $S[\Lambda] : \psi(x) \to S[\Lambda]\psi(\Lambda^{-1}x)$, that corresponds to the vector Lorentz transformation matrix $\Lambda = \Lambda^\mu_\nu$.

(a) (15 pt) Prove the following property of $S[\Lambda]$

$$S^{-1}[\Lambda]\gamma^\mu S[\Lambda] = \Lambda^\mu_\nu \gamma^\nu \tag{8}$$

Hint: the simplest way is to consider the Lorentz transformation of the Dirac equation, $(i\gamma^\mu\partial_\mu - m)\psi = 0$, and use the fact that both the derivative and mass terms should transform by the same law. Another way is to use the explicit form of $S[\Lambda]$

$$S[\Lambda] = e^{\frac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta}},$$

and apply differentiation with respect to $\Omega_{\alpha\beta}$ (Feynman trick) or another techniques to evaluate $S^{-1}[\Lambda]\gamma^\mu S[\Lambda]$ directly.

(b) (15 pt) Show that the spinor structures $\bar\psi\gamma^\mu\psi$ and $\bar\psi\gamma^\mu\gamma^\nu\psi$ transform like a Lorentz vector and tensor respectively, i.e.

$$\bar\psi\gamma^\mu\psi \to \Lambda^\mu_\nu \bar\psi\gamma^\nu\psi, \quad \bar\psi\gamma^\mu\gamma^\nu\psi \to \Lambda^\mu_\alpha\Lambda^\nu_\beta \bar\psi\gamma^\alpha\gamma^\beta\psi, \tag{9}$$

under the Lorentz transformation.

4. Parity and charge conjugation (20 points)

Obtain how each spinor bilinear transforms under the respective discrete symmetry and fill in the table

Discrete symmetry	$\bar\psi\psi$	$i\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$i\bar\psi[\gamma^\mu,\gamma^\nu]\psi$
$\hat P$					
$\hat C$					
$\hat C\hat P$					

$\hat C\hat P$ means the combination symmetry, when one applies parity transformation and after that the charge conjugation or vice versa.

Show that every bilinear form given in the table is hermitian. Why is it important to satisfy the hermiticity of the Lagrangian terms?

BONUS: How to define parity transformation in 2+1 dimensions (important e.g. in graphene physics)?

5. Properties of chiral projectors (10 points)

Consider the chiral projectors $P_{L,R} \equiv (\mathbb{1}_{4\times4} \mp \gamma^5)/2$.

(a) (5 pt) Show that $P_{L,R}$ satisfy the geneal properties of projectors:

$$P^2_{L,R} = P_{L,R}, \quad P_R + P_L = \mathbb{1}_{4\times4}, \quad P_R P_L = P_L P_R = 0. \tag{10}$$

(b) (5 pt) Express ψ as $P_L\psi + P_R\psi \equiv \psi_L + \psi_R$ and find, what chiral spinor bilinears one gets from those, listed in the table, and fill in the empty cells.

Bilinear form	$\bar\psi\psi$	$i\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$i\bar\psi[\gamma^\mu,\gamma^\nu]\psi$
Chiral structure	$\bar\psi_L\psi_R + \bar\psi_R\psi_L$				