Relativistic QFT (Theo 6a): Exercise Sheet 11
Total: 100 points

29/01/2021

1. High-energy Møller scattering in the helicity formalism (50 points)

Consider the elastic scattering of two electrons (Møller scattering) in Weyl representation. In the ultrarelativistic limit the Dirac equation can be diagonalized, and Weyl spinors take the form

$$u_\uparrow = \sqrt{2E}\begin{pmatrix}0\\ \varepsilon_\uparrow\end{pmatrix}, \quad u_\downarrow = \sqrt{2E}\begin{pmatrix}\varepsilon_\downarrow\\ 0\end{pmatrix}, \quad \text{where} \quad \varepsilon_\uparrow = \begin{pmatrix}\cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\end{pmatrix}, \quad \varepsilon_\downarrow = \begin{pmatrix}-e^{-i\phi}\sin(\theta/2)\\ \cos(\theta/2)\end{pmatrix}. \tag{1}$$

(a) Taking the initial 3-momentum $\vec{p}$ of the first particle along z-axis and working in the c.o.m. frame, find the full set of the initial and final helicity spinors, i.e. the exact form of the spinors (10 points)

$$\varepsilon_{\uparrow(\downarrow)}(p), \quad \varepsilon_{\uparrow(\downarrow)}(k), \quad \varepsilon_{\uparrow(\downarrow)}(p'), \quad \varepsilon_{\uparrow(\downarrow)}(k'), \tag{2}$$

where p' is the final momentum of the first electron, k and k' are the initial and the final momenta of the second electron.

(b) Calculate the exact forms of the all possible helicity combinations (20 points)

$$\text{t-channel:} \quad \bar{u}_{h_{p'}}(p')\gamma^\mu u_{h_p}(p), \quad \bar{u}_{h_{k'}}(k')\gamma^\mu u_{h_k}(k), \tag{3}$$

$$\text{u-channel:} \quad \bar{u}_{h_{k'}}(k')\gamma^\mu u_{h_p}(p), \quad \bar{u}_{h_{p'}}(p')\gamma^\mu u_{h_k}(k), \tag{4}$$

where the helicities h_p, $h_{p'}$, h_k and $h_{k'}$ can be $\uparrow$ ("+") or $\downarrow$ ("−"). Show that in case when the helicities are not conserved, the combinations equal zero.

(c) Compute the full set of the helicity amplitudes, i.e. the amplitudes (20 points)

$$\text{t-channel:} \quad \mathcal{M}^t_{++,++}, \quad \mathcal{M}^t_{--,--}, \quad \mathcal{M}^t_{+-,+-}, \quad \mathcal{M}^t_{-+,-+}, \tag{5}$$

$$\text{u-channel:} \quad \mathcal{M}^u_{++,++}, \quad \mathcal{M}^u_{--,--}, \quad \mathcal{M}^u_{-+,+-}, \quad \mathcal{M}^u_{+-,-+}. \tag{6}$$

Show that averaging over the initial and summing over the final helicities gives the same result as with the trace technique obtained in Lecture 21, i.e.

$$\frac{d\sigma}{d\Omega^{cm}} = \frac{1}{64\pi^2 s}\,\frac{1}{4}\sum_{h_p,\,h_{p'}\,h_k\,h_{k'}} |\mathcal{M}_{h_p h_{p'},\,h_k h_{k'}}|^2 = \frac{\alpha^2}{s\sin^4\theta}\left(3+\cos^2\theta\right)^2, \tag{7}$$

where $\alpha = e^2/4\pi$.

2. Electron-nucleon scattering (50 points)

Consider the scattering of the electron with mass m on the heavy nucleon with mass M ($M \gg m$) in the Breit frame, where the initial and final momenta of the nucleon are defined as

$$p^\mu = \left(P^0, 0, 0, -\frac{\Delta}{2}\right), \quad p'^\mu = \left(P^0, 0, 0, \frac{\Delta}{2}\right), \quad (p'-p)^\mu = \Delta^\mu, \quad P^0 = \sqrt{M^2 - \frac{t}{4}}, \tag{8}$$

and the same for the electron:

$$k^\mu = E_B(1, \sin\theta, 0, \cos\theta), \quad k'^\mu = E_B(1, \sin\theta, 0, -\cos\theta), \quad 2E_B\cos\theta = \Delta. \tag{9}$$

Note that the the angle so defined is not the scattering angle, i.e., change of direction of the final electron momentum with respect to the initial one. The scattering angle θ_e is obtained from $\cos\theta_e = \hat{k}\cdot\hat{k}' = -\cos 2\theta$, or $\theta = \pi/2 - \theta_e/2$.

The nucleon spinors in the Dirac representation are

$$u_\lambda(p) = \sqrt{P^0+M}\begin{pmatrix}\varepsilon_\lambda \\ -\lambda\frac{\Delta/2}{P^0+M}\varepsilon_\lambda\end{pmatrix}, \quad u_{\lambda'}(p') = \sqrt{P^0+M}\begin{pmatrix}\varepsilon_{\lambda'} \\ \lambda'\frac{\Delta/2}{P^0+M}\varepsilon_{\lambda'}\end{pmatrix}, \tag{10}$$

$$\text{where} \quad \varepsilon_{\lambda(\lambda')=\uparrow} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \varepsilon_{\lambda(\lambda')=\downarrow} = \begin{pmatrix}0\\1\end{pmatrix}, \quad \lambda = \pm 1, \tag{11}$$

and the electron spinors are

$$u_h(k) = \sqrt{E_B}\begin{pmatrix}\varepsilon_h \\ 2h\varepsilon_h\end{pmatrix}, \quad u_{h'}(k') = \sqrt{E_B}\begin{pmatrix}\varepsilon_{h'} \\ 2h'\varepsilon_{h'}\end{pmatrix}, \tag{12}$$

where

$$\varepsilon_{h=\uparrow} = \begin{pmatrix}\cos\theta/2\\ \sin\theta/2\end{pmatrix}, \quad \varepsilon_{h=\downarrow} = \begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix}, \quad \varepsilon_{h'=\uparrow} = \begin{pmatrix}\sin\theta/2\\ \cos\theta/2\end{pmatrix}, \quad \varepsilon_{h'=\downarrow} = \begin{pmatrix}-\cos\theta/2\\ \sin\theta/2\end{pmatrix}, \tag{13}$$

Consider the expression for the helicity amplitudes

$$\begin{aligned}\mathcal{M}^{h'h}_{\lambda'\lambda} &= -\frac{e^2}{t}\bar{u}_{h'}(k')\gamma^\mu u_h(k)\,\bar{u}_{\lambda'}(p')\gamma_\mu u_\lambda(p) \\ &= -\frac{e^2}{t}\bar{u}_{h'}(k')\gamma^\mu u_h(k)\,\bar{u}_{\lambda'}(p')\left[\frac{(p+p')_\mu}{2M} + \frac{i\sigma_{\mu\alpha}\Delta^\alpha}{2M}\right]u_\lambda(p),\end{aligned} \tag{14}$$

where Gordon's identity for the nucleon part was used in the second line.

(a) Obtain the expressions for all non-vanishing helicity amplitudes (25 points)
Remember that the helicity of the electron is conserved, $h = h'$). To evaluate the contribution of the magnetic term $\sim \sigma_{\mu\alpha}$ use spin raising/lowering operators $\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma^1 \pm i\sigma^2)$, see Lecture 22 for further details. Note that the scalar product of any two vectors $\vec{a} = (a^1, a^2, a^3)$ and $\vec{b} = (b^1, b^2, b^3)$ is written via the $\pm$ components as $\vec{a}\cdot\vec{b} = a^+b^- + a^-b^+ + a^3b^3$.

(b) Compare the results for helicity amplitudes to those for electron scattering off a spin-0 proton (10 points)
The latter can be easily obtained from the result for the electric term (first term in the square bracket) in Eq. (11):

$$\mathcal{M}^{h'h} = -\frac{e^2}{t}\bar{u}_{h'}(k')\gamma^\mu u_h(k)\,(p+p')_\mu. \tag{15}$$

(c) Square the helicity amplitudes, sum over final polarizations and average over the initial ones, and compare results for the spin-0 and spin-1/2 proton. To compare with the result obtained in the c.m. frame in Lecture 22,

$$\frac{1}{4}\sum_{h,h',\lambda,\lambda'}|\mathcal{M}|^2 = \frac{se^4}{E_{cm}^2\sin^4\frac{\theta_e^{cm}}{2}}\left[\cos^2\frac{\theta_e^{cm}}{2} - \frac{t}{2s}\sin^2\frac{\theta_e^{cm}}{2}\right] \tag{16}$$

for spin-1/2 proton, while only the first term in the square bracket survives in the spin-0 case. Use the invariants t, s expressed in Breit and c.m. frames, e.g., $s = M^2 + 2\sqrt{s}E_{cm}$ and $t = -4E_{cm}^2\sin^2\frac{\theta_e^{cm}}{2}$ (work out the respective expressions in Breit frame). (15 points)