Relativistic QFT (Theo 6a): Exercise Sheet 10
Total: 100 points

22/01/2021

1. Traces and identities of the Dirac matrices (40 points)

(a) Prove the following relations:

$$\mathrm{Tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n+1}}\right) = 0 \tag{1}$$

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4i\epsilon^{\mu\nu\rho\sigma} \tag{2}$$

$$\begin{aligned}
&\mathrm{Tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}\gamma^{\mu_6}\right) = \\
&\quad 4(g^{\mu_1\mu_2}g^{\mu_3\mu_4}g^{\mu_5\mu_6} + g^{\mu_1\mu_2}g^{\mu_3\mu_6}g^{\mu_4\mu_5} - g^{\mu_1\mu_2}g^{\mu_3\mu_5}g^{\mu_4\mu_6} \\
&\quad - g^{\mu_1\mu_3}g^{\mu_2\mu_4}g^{\mu_5\mu_6} - g^{\mu_1\mu_3}g^{\mu_2\mu_6}g^{\mu_4\mu_5} + g^{\mu_1\mu_3}g^{\mu_2\mu_5}g^{\mu_4\mu_6} \\
&\quad + g^{\mu_1\mu_4}g^{\mu_2\mu_3}g^{\mu_5\mu_6} + g^{\mu_1\mu_4}g^{\mu_2\mu_6}g^{\mu_3\mu_5} - g^{\mu_1\mu_4}g^{\mu_2\mu_5}g^{\mu_3\mu_6} \\
&\quad - g^{\mu_1\mu_5}g^{\mu_2\mu_3}g^{\mu_4\mu_6} - g^{\mu_1\mu_5}g^{\mu_2\mu_6}g^{\mu_3\mu_4} + g^{\mu_1\mu_5}g^{\mu_2\mu_4}g^{\mu_3\mu_6} \\
&\quad + g^{\mu_1\mu_6}g^{\mu_2\mu_3}g^{\mu_4\mu_5} + g^{\mu_1\mu_6}g^{\mu_2\mu_5}g^{\mu_3\mu_4} - g^{\mu_1\mu_6}g^{\mu_2\mu_4}g^{\mu_3\mu_5})
\end{aligned} \tag{3}$$

Hint: use twice the decomposition of the product of 3 γ-matrices,

$$\gamma^\mu\gamma^\nu\gamma^\alpha = S^{\mu\nu\alpha\beta}\gamma_\beta + i\epsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\beta, \quad S^{\mu\nu\alpha\beta} = g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} \tag{4}$$

$$\begin{aligned}
\epsilon^{\mu\nu\sigma\rho}\epsilon^{\alpha\beta\gamma}{}_\rho &= -det\begin{vmatrix} g^{\mu\alpha} & g^{\mu\beta} & g^{\mu\gamma} \\ g^{\nu\alpha} & g^{\nu\beta} & g^{\nu\gamma} \\ g^{\sigma\alpha} & g^{\sigma\beta} & g^{\sigma\gamma} \end{vmatrix} \\
&= -g^{\mu\alpha}(g^{\nu\beta}g^{\sigma\gamma} - g^{\nu\gamma}g^{\sigma\beta}) + g^{\nu\alpha}(g^{\mu\beta}g^{\sigma\gamma} - g^{\mu\gamma}g^{\sigma\beta}) - g^{\sigma\alpha}(g^{\mu\beta}g^{\nu\gamma} - g^{\mu\gamma}g^{\nu\beta}).
\end{aligned} \tag{5}$$

(b) Prove the following identities:

$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = 4g^{\alpha\beta} \tag{6}$$

$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\beta\gamma^\alpha \tag{7}$$

$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\rho\gamma_\mu = 2\left(\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^\sigma + \gamma^\sigma\gamma^\beta\gamma^\alpha\gamma^\rho\right) \tag{8}$$

(c) Using the Dirac equation for u-spinors

$$(\not{p} - m)u(p) = 0, \quad \bar{u}(p)(\not{p} - m) = 0, \tag{9}$$

prove the Gordon identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\Big[\frac{p^\mu + p'^\mu}{2m} + i\sigma^{\mu\nu}\frac{p'_\nu - p_\nu}{2m}\Big]u(p), \tag{10}$$

where $\sigma^{\mu\nu} \equiv i[\gamma^\mu, \gamma^\nu]/2$.

2. Compton scattering in QED (50 points)

Consider the Compton scattering process ($e\gamma \to e\gamma$) at the tree level in QED.

(a) Write down the expressions for the invariant amplitudes A_s and A_u which correspond to s- and u-channel diagram.

(b) Averaging over the initial spin states and sum over the final spin states of the electron and photon, applying formulae

$$\sum_{\lambda=-1,1} \epsilon_\mu^\lambda(q)\epsilon_\nu^{*\lambda}(q) = -g_{\mu\nu}, \qquad \sum_{h=-\frac{1}{2},\frac{1}{2}} u(p)_h \bar{u}(p)_h = \not{p} + m \tag{11}$$

for the spin sums of the photon polarization vectors ϵ_μ^λ and the electron spinors u_h, show that the averaged squared modulus can be expressed as

$$\overline{|A_s + A_u|}^2 = f(s,u) + g(s,u) + f(u,s) + g(u,s), \tag{12}$$

where functions f and g are defined as

$$f(s,u) \equiv |A_s|^2 = \frac{\mathrm{Tr}\Big[(\not{p}' + m)\gamma^\mu(\not{p} + \not{q} + m)\gamma^\nu(\not{p} + m)\gamma_\nu(\not{p} + \not{q} + m)\gamma_\mu\Big]}{4(s-m^2)^2}, \tag{13}$$

$$g(s,u) \equiv A_s A_u^\dagger = \frac{\mathrm{Tr}\Big[(\not{p}' + m)\gamma^\mu(\not{p} + \not{q} + m)\gamma^\nu(\not{p} + m)\gamma_\mu(\not{p} - \not{q}' + m)\gamma_\nu\Big]}{4(s-m^2)(u-m^2)}. \tag{14}$$

Here q and q' are initial and final photon momenta, p and p' are initial and final electron momenta respectively. Using trace technique, obtain $\overline{|A_s + A_u|}^2$ in terms of invariants.

(c) Obtain the differential cross section $d\sigma/dt$ in terms of Mandelstam invariants and compare the result with the one obtained in the scalar QED. Find the low- and high-energy behavior of the differential cross section.

3. Plane wave solution of the Dirac equation in Dirac representation (10 points)

Consider the γ-matrices in the Dirac representation $\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$, $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$. The positive and negative energy plane wave solutions of the Dirac equation can be written as

$$\psi^{(E>0)} = u(p)e^{-ipx}, \quad \psi^{(E<0)} = v(p)e^{ipx} \tag{15}$$

respectively as well.

(a) Pasting (15) into the Dirac equation $(i\not{\partial} - m)\psi$ show that the spinors $u(p)$ and $v(p)$ can be expressed as

$$u(p) = \sqrt{E+m}\begin{pmatrix} \xi \\ \frac{\vec{\sigma}\vec{p}}{E+m}\xi \end{pmatrix}, \quad v(p) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\vec{p}}{E+m}\eta \\ \eta \end{pmatrix}, \tag{16}$$

where ξ and η are 2×2 spinors and σ^i Pauli matrices.

(b) Knowing that the ξ and η spinors with definite helicities h have the following properties

$$\xi_{h'}^\dagger \xi_h = \delta_{h'h}, \quad \eta_{h'}^\dagger \eta_h = \delta_{h'h}, \tag{17}$$

either in Dirac representation, show that u and v spinors satisfy the relations

$$\bar{u}_{h'}u_h = 2m\delta_{h'h}, \quad u_{h'}^\dagger u_h = 2E\delta_{h'h}, \quad \bar{v}_{h'}v_h = -2m\delta_{h'h}, \quad v_{h'}^\dagger v_h = 2E\delta_{h'h}, \tag{18}$$

$$\bar{u}_{h'}v_h = 0, \quad u_{h'}^\dagger(\vec{p})v_h(-\vec{p}) = 0. \tag{19}$$