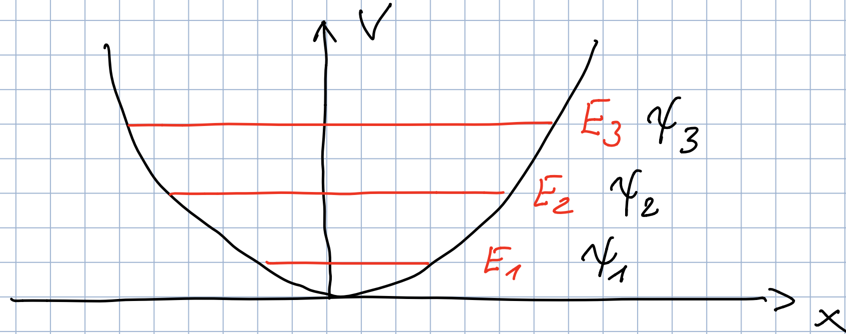
⟹ VORLESUNG 9 QM

KAPITEL 3: FORMALISMUS DER QM

↳ 3.1 HILBERT RAUM

H.O. $V(x) = \frac{1}{2} m \omega^2 x^2$



$$E_n = \hbar \omega \left(n + \frac{1}{2} \right)$$

$n = 0, 1, 2, \ldots$

$$\psi(x) = \sum_n c_n \psi_n(x)$$

$|c_n|^2$

↓ ↓

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$

ZUSTANDS

VEKTOR IM HILBERT RAUM

$\psi_n(x)$ QUADRATISCH INTEGRIERBARE FUNKTION ÜBER $[a, b]$

H.O. $[-\infty, +\infty]$

$$\int_a^b dx \; |\psi_n(x)|^2 < \infty$$

SATZ ALLER Q.I. FUNKTIONEN BILDET VEKTOR RAUM (HILBERT RAUM)

(∞ DIM!)

INNERE PRODUKT $|f\rangle$, $|g\rangle$

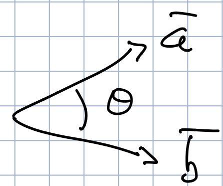
$$\langle f | g \rangle \equiv \int_a^b dx \; f^*(x) \, g(x)$$

$$\langle f | f \rangle = \int_a^b dx \; f^*(x) \, f(x) < \infty$$

$$\langle g | g \rangle = \int_a^b dx \; g^*(x) \, g(x) < \infty$$

SCHWARZ UNGLEICHEIT

$$|\bar{a}\cdot\bar{b}|^2 \leq |\bar{a}|^2 \, |\bar{b}|^2$$



$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle \, \langle g|g\rangle$$

EIGENSCHAFTEN

$\hookrightarrow$
$$\langle g|f\rangle = \int_a^b dx \; g^*(x)\, f(x)$$
$$= \left(\int_a^b dx \; g(x)\, f^*(x) \right)^*$$
$$= \left(\underbrace{\int_a^b dx \; f^*(x)\, g(x)} \right)^*$$
$$= \langle f|g\rangle^*$$

$\hookrightarrow$
$$\langle f|f\rangle = \int_a^b dx \; |f(x)|^2 \in \mathbb{R}$$

$$\langle f|f\rangle = 0 \iff f(x) = 0$$

- ORTHOGONALITÄT, NORMIERBARKEIT

$$\{\psi_n(x)\}$$

$$|\psi_n\rangle$$

$$\rightsquigarrow \quad \langle \psi_n | \psi_m \rangle = \delta_{nm}$$

$$\int_a^b dx \; \psi_n^*(x)\, \psi_m(x) = \delta_{nm}$$

- VOLLSTÄNDIGKEIT (COMPLETENESS)

$$\rightsquigarrow \quad \psi(x) = \sum_n c_n \, \psi_n(x)$$

$$|\psi\rangle = \sum_n c_n \, |\psi_n\rangle$$

$$c_n = \langle \psi_n | \psi \rangle$$

$$= \int_a^b dx \; \psi_n^*(x)\, \psi(x)$$

↳ 3.2 OBSERVABLEN

• HERMITISCHE OPERATOREN

HERMITISCHE OPERATOREN
IM HILBERT RAUM

$$\hat{Q} = \hat{Q}^{\dagger}$$

$$\langle \hat{Q} \rangle = \int_a^b dx \, \underset{\uparrow \atop f^*}{\psi^*(x)} \, \underbrace{\hat{Q} \, \psi(x)}_{g}$$

$$= \langle \psi | \hat{Q} \psi \rangle$$

$$\langle \hat{Q} \rangle = \langle \hat{Q} \rangle^* \qquad \in \mathbb{R}$$

$$\downarrow$$

$$\langle \psi | \hat{Q} \psi \rangle = \langle \psi | \hat{Q} \psi \rangle^*$$

$$= \langle \hat{Q} \psi | \psi \rangle \qquad \underline{\underline{\forall \psi}}$$

$$\hat{Q} = \hat{Q}^{\dagger}$$

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle \qquad \underline{\underline{\forall f, g}}$$

$\hat{Q}$ HERMITISCH

BEISPIEL $\quad \hat{Q} = \hat{p} = -i\hbar \frac{d}{dx}$

$$\langle f | \hat{p}\, g \rangle = \langle \hat{p} f | g \rangle$$

$$= \int_{-\infty}^{+\infty} dx \; f^*(x)\, \hat{p}\, g(x)$$

$$= \int_{-\infty}^{+\infty} dx \; f^*(x) \left(-i\hbar \frac{d}{dx} g\right)$$

$$= \cancel{-i\hbar\, f^* g \Big|_{-\infty}^{+\infty}} + i\hbar \int_{-\infty}^{+\infty} dx \left(\frac{df}{dx}\right)^* g$$

$$= \int_{-\infty}^{+\infty} dx \left(-i\hbar \frac{d}{dx} f\right)^* g$$

$$= \int_{-\infty}^{+\infty} dx \left(\hat{p} f\right)^* g$$

$$= \langle \hat{p} f | g \rangle \qquad \forall\, f, g$$

$-i\hbar \frac{d}{dx}$ IST HERMITISCH

$\frac{d}{dx}$ IST NICHT!

- WOHL DEFINIERTE ZUSTÄNDE / EIGENZUSTÄNDE

$$\psi_n$$

$$\hat{H} \curvearrowright E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

$$\langle \hat{H} \rangle = \langle \underbrace{\psi_n | \hat{H} \psi_n}_{\text{EIGENZUSTÄNDE}} \rangle = \underbrace{E_n}_{\text{EIGENWERTE}}$$

σ IST 0 !

$$| \hat{Q} \psi \rangle = q | \psi \rangle \qquad q \in \mathbb{R}$$

$$\rightsquigarrow \langle \hat{Q} \rangle = \langle \psi | \hat{Q} \psi \rangle = q \underbrace{\langle \psi | \psi \rangle}_{1}$$

$$\sigma^2 = \langle (\hat{Q} - \langle \hat{Q} \rangle)^2 \rangle$$

$$= \langle \underbrace{(\hat{Q} - \langle \hat{Q} \rangle)(\hat{Q} - \langle \hat{Q} \rangle)} \rangle$$

$$= \langle \underbrace{\psi}_{f} | (\hat{Q} - \langle \hat{Q} \rangle) \underbrace{(\hat{Q} - \langle \hat{Q} \rangle) \psi}_{g} \rangle$$

$\hat{Q}$ HERMITISCH

$\hat{Q} - \underbrace{\langle \hat{Q} \rangle}_{\in \mathbb{R}}$ HERMITISCH

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle$$

$$\sigma^2 = \langle (\hat{Q} - \langle \hat{Q} \rangle)\psi \,|\, (\hat{Q} - \langle \hat{Q} \rangle)\psi \rangle$$

$$|(\hat{Q} - \langle \hat{Q} \rangle)\psi \rangle$$

$$= \underbrace{|\hat{Q}\psi\rangle} - \langle \hat{Q} \rangle |\psi\rangle$$

$$q|\psi\rangle - q|\psi\rangle$$

$$= 0$$

$$\sigma^2 \overset{!}{=} 0$$ (SCHARFE ZUSTÄNDE).

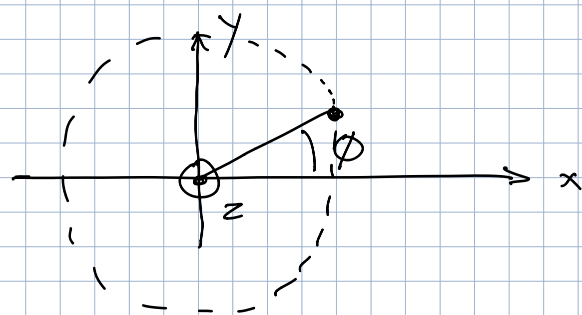
ψ_n

$$\langle \hat{H} \rangle = E_n \overset{H.O.}{=} \hbar\omega\left(n + \frac{1}{2}\right)$$

$n = 0, 1, \dots, \infty$

$\{E_n\}$ SPEKTRUM.

- **BEISPIEL** $\qquad \hat{Q} = i \dfrac{\partial}{\partial \phi}$



$$\phi \in [0, 2\pi]$$

$$f(\phi) = f(\phi + 2\pi)$$

$$\int_0^{2\pi} d\phi \; |f(\phi)|^2 < \infty$$

$\hat{Q}$ IST HERMITISCH

$$\langle f | \hat{Q} g \rangle$$

$$= \int_0^{2\pi} d\phi \; f^*(\phi) \; i \frac{\partial}{\partial \phi} g(\phi)$$

$$= \cancel{i f^* g \Big|_0^{2\pi}} - i \int_0^{2\pi} d\phi \; \frac{\partial f^*}{\partial \phi} \; g$$

$$= \int_0^{2\pi} d\phi \left(i \frac{\partial}{\partial \phi} f \right)^* g$$

$$= \langle \hat{Q} f \mid g \rangle \qquad \forall f, g$$

$\hat{Q}$ IST HERMITISCH !

$$\langle f \mid g \rangle \equiv \int_0^{2\pi} d\phi \; f^*(\phi) \, g(\phi)$$

$$\langle \hat{Q} f \mid g \rangle \equiv \int_0^{2\pi} d\phi \; (\hat{Q} f)^* \, g$$

EIGENWERTE VON $\hat{Q} = i \frac{\partial}{\partial \phi}$

$$| \hat{Q} f \rangle = q | f \rangle$$

$$\downarrow$$

$$i \frac{\partial}{\partial \phi} f(\phi) = q \, f(\phi)$$

$$\frac{\partial f}{\partial \phi} = - i q \, f$$

$$f(\phi) = C\, e^{-iq\phi}$$

EIGENFUNKTION

$$f(\phi) = f(\phi + 2\pi)$$

$$e^{-iq\phi} = e^{-iq(\phi + 2\pi)}$$

$$e^{-iq\,2\pi} = \underline{1}$$

$$\Longrightarrow \quad q = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots$$

EIGENWERTE $\Rightarrow$ SPEKRUM VON $\hat{Q}$

↳ 3.3 EIGENZUSTÄNDE EINES HERMITISCHEN OPERATORS

SPEKTRUM → DISKRET

SPEKTRUM → KONTINUIERLICH

- DISKRET

$$\longrightarrow \quad |\hat{Q} f\rangle = q\,|f\rangle$$

$$\langle \hat{Q} \rangle = \langle f | \hat{Q} f \rangle$$

$$= q \langle f | f \rangle$$

$$= \langle \hat{Q} f | f \rangle$$

$$= q^* \langle f | f \rangle$$

$$q = q^* \qquad \in \mathbb{R}$$

EIGENWERTE REEL

→ EIGENZUSTÄNDE

$$|\hat{Q} f\rangle = q\,|f\rangle \qquad (1)$$

$$|\hat{Q} g\rangle = q'\,|f\rangle \qquad (2)$$

$$\langle f|\hat{Q} g\rangle \underset{(2)}{=} q'\,\langle f|g\rangle$$

$\hat{Q} = \hat{Q}^+$ $\qquad \|$

$$\langle \underset{\sim}{\hat{Q} f} | g\rangle \underset{(1)}{=} q\,\langle f|g\rangle$$

$\underset{\sim}{f \neq g} \qquad \underline{\underline{q \neq q'}} \quad \Rightarrow \langle f|g\rangle = 0$

ORTHOGONAL !

$$|\psi\rangle = \sum_n c_n\,|\psi_n\rangle$$

HILBERT (∞ DIM)

<u>AXIOM</u> EIGENFUNKTIONEN
EINER OBSERVABLE
BILDEN VOLLSTÄNDIGEN SATZ

- KONTINUIERLICHE SPEKTRUM

$$\hat{p} = -i\hbar \frac{d}{dx}$$

$$\hat{p}\, f_p(x) = p\, f_p(x)$$

$f_p(x)$: EIGENFUNKTION; p: EIGENWERT

$$-i\hbar \frac{d}{dx} f_p = p\, f_p$$

$$\frac{df_p}{dx} = \frac{i}{\hbar} p\, f_p$$

$$f_p(x) = C\, e^{\frac{i}{\hbar} p \cdot x}$$

} NICHT QUADRATISCH INTERGRIERBAR

$$\langle f_{p'} | f_p \rangle = \int_{-\infty}^{+\infty} dx\, f_{p'}^{*}(x)\, f_p(x)$$

$p' \neq p$

$$= |C|^2 \int_{-\infty}^{+\infty} dx\, e^{\frac{i}{\hbar}(p-p')\cdot x}$$

$$= |C|^2 \; 2\pi \, \delta\left(\frac{1}{\hbar}(p-p')\right)$$

$$\int dx \; e^{iax} = 2\pi\,\delta(a)$$

$$\delta(cx) = \frac{1}{c}\,\delta(x)$$

$$= |C|^2 \cdot 2\pi\hbar \;\; \delta(p-p')$$

<u>WAHL</u> $\quad C = \frac{1}{\sqrt{2\pi\hbar}}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \, e^{\frac{i}{\hbar} p \cdot x}$$

$$\langle f_{p'} | f_p \rangle = \delta(p-p')$$

DIRAC ORTHONORMALITÄT

$$\Updownarrow$$

($\delta_{pp'}$ DISKRET)

$$f(x) \Longleftrightarrow \sum_p c_p \; f_p(x) \qquad \text{DISKRET}$$

$$\Downarrow$$

$$= \int_{-\infty}^{+\infty} dp \; c(p) \; f_p(x)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \; c(p) \, e^{\frac{i}{\hbar} p \cdot x}$$

(F. T.)

$$\langle f_{p'} | f \rangle = \int_{-\infty}^{+\infty} dp \, c(p) \underbrace{\int dx \, f_{p'}^{*}(x) \; f_p(x)}_{\delta(p'-p)}$$

$$= c(p')$$

$$c(p) = \langle f_p | f \rangle$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx \;\; e^{-\frac{i}{\hbar} p x} \; f(x)$$

WELLENFUNKTION
IM IMPULSRAUM