$\Rightarrow$ VORLESUNG 12 QM

H.O $\quad \hat{H} = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$

KOHÄRENTER ZUSTAND $\quad a \, |\underline{\underline{\alpha}}\rangle = \alpha \, |\alpha\rangle$

$\alpha \in \mathbb{C}$

$$\hat{H} \, |n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, |n\rangle$$

$$\begin{cases} a = \dfrac{1}{\sqrt{2}} \left(\hat{X} + i \hat{\underline{P}} \right) \\[2ex] a^\dagger = \dfrac{1}{\sqrt{2}} \left(\hat{X} - i \hat{\underline{P}} \right) \end{cases}$$

$$\begin{cases} \hat{X} \equiv \sqrt{\dfrac{m\omega}{\hbar}} \; \hat{x} \\[2ex] \hat{\underline{P}} \equiv \sqrt{\dfrac{1}{\hbar\omega m}} \; \hat{p} \end{cases}$$

$$\langle \hat{H} \rangle = \langle \alpha | \hat{H} | \alpha \rangle$$

$$= \hbar\omega \left(|\alpha|^2 + \frac{1}{2} \right)$$

$\uparrow$

MITTLERE ANREGUNGSNIVEAU

$$\langle \hat{X} \rangle = \sqrt{2}\ \mathrm{Re}\,\alpha$$

$$\langle \hat{P} \rangle = \sqrt{2}\ \mathrm{Im}\,\alpha$$

$$\sigma_X^2 = \langle (\hat{X} - \langle \hat{X} \rangle)^2 \rangle$$

$$= \langle \alpha | \hat{X}^2 | \alpha \rangle - \langle \hat{X} \rangle^2$$

$$\Big\downarrow \qquad \hat{X} = \frac{1}{\sqrt{2}}(a + a^+)$$

$$= \frac{1}{2} \langle \alpha | (a + a^+)(a + a^+) | \alpha \rangle - 2(\mathrm{Re}\,\alpha)^2$$

$$= \frac{1}{2} \langle a^2 + (a^+)^2 + a a^+ + a^+ a \rangle - 2(\mathrm{Re}\,\alpha)^2$$

$$[a, a^+] = 1$$

$$a a^+ = a^+ a + 1$$

$$\sigma_X^2 = \frac{1}{2}\langle a^2 + (a^\dagger)^2 + 2a^\dagger a + 1\rangle - 2(\mathrm{Re}\,\alpha)^2$$

$$\langle\quad\rangle = \langle\alpha|\qquad|\alpha\rangle$$

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle\alpha|a^\dagger = \alpha^*\langle\alpha|$$

$$= \frac{1}{2}\left(\underbrace{\alpha^2 + (\alpha^*)^2 + 2\alpha^*\alpha} + 1\right) - 2(\mathrm{Re}\,\alpha)^2$$

$$= \frac{1}{2}\left(\underbrace{(\alpha + \alpha^*)}_{2\mathrm{Re}\,\alpha}{}^2 + 1\right) - 2(\mathrm{Re}\,\alpha)^2$$

$$= \cancel{2(\mathrm{Re}\,\alpha)^2} + \frac{1}{2} - \cancel{2(\mathrm{Re}\,\alpha)^2}$$

$$\sigma_X^2 = \frac{1}{2} \iff \sigma_X = \frac{1}{\sqrt{2}} \longleftarrow$$

$$\longrightarrow \sigma_x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_P^2 = \langle \hat{P}^2 \rangle - \langle \hat{P} \rangle^2$$

$$\downarrow \qquad \hat{P} = \frac{1}{i\sqrt{2}} \left(a - a^{+} \right)$$

$$= \frac{1}{2} \left(\Big(\underbrace{\frac{\alpha - \alpha^*}{i}}_{2\,\mathrm{Im}\,\alpha} \Big)^2 + 1 \right) - 2 \left(\mathrm{Im}\,\alpha \right)^2$$

$$\sigma_P^2 = \frac{1}{2} \iff \boxed{\sigma_P = \frac{1}{\sqrt{2}}}$$

$$\longrightarrow \quad \boxed{\sigma_p = \sqrt{\frac{\hbar \omega m}{2}}}$$

$$\sigma_x \, \sigma_p = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{\hbar m \omega}{2}} = \frac{\hbar}{2}$$

MINIMALE
UNSCHÄRFE

↳ WELLENFUNKTION IN X

$$\langle x | \alpha \rangle \equiv \psi_\alpha(x)$$

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$\langle x|$

$$\langle x|a|\alpha\rangle = \alpha \underbrace{\langle x|\alpha\rangle}_{\psi_\alpha(x)}$$

$$\frac{1}{\sqrt{2}}\langle x|\hat{X} + i\hat{P}|\alpha\rangle = \alpha\,\psi_\alpha(x)$$

$$\hat{P} = -i\frac{d}{dX}$$

$$\frac{1}{\sqrt{2}}\left(X+\frac{d}{dX}\right)\underbrace{\langle X|\alpha\rangle}_{\psi_\alpha(X)} = \alpha\,\psi_\alpha(x)$$

$$\frac{d\psi_\alpha}{dX} = -\left(X-\sqrt{2}\alpha\right)\psi_\alpha$$

$$\psi_\alpha(X) = C\, e^{-\frac{1}{2}\left(X-\sqrt{2}\alpha\right)^2}$$

↳ WELLENFUNKTION IN P

$$\langle p | \alpha \rangle = \varphi_\alpha(p)$$

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle p|a|\alpha\rangle = \alpha\langle p|\alpha\rangle$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}\langle p|\left(\hat{X} + i\hat{P}\right)|\alpha\rangle = \alpha\langle p|\alpha\rangle$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}\left(+i\frac{d}{dP} + iP\right)\underbrace{\langle p|\alpha\rangle}_{\varphi_\alpha(p)} = \alpha\,\varphi_\alpha(p)$$

$$\left(\frac{d}{dP} + P\right)\varphi_\alpha(p) = -i\sqrt{2}\alpha\,\varphi_\alpha(p)$$

$$\frac{d\varphi_\alpha}{dP} = -\left(P + i\sqrt{2}\alpha\right)\varphi_\alpha(p)$$

$$\varphi_\alpha(P) = C'\, e^{-\frac{1}{2}\left(P + i\sqrt{2}\alpha\right)^2}$$

$$\hat{P}\,|p\rangle = p\,|p\rangle$$

$$\langle x|\hat{P}|p\rangle = p\,\langle x|p\rangle$$

$$\downarrow$$

$$-i\hbar\frac{d}{dx}\langle x|p\rangle = p\,\langle x|p\rangle$$

$$\langle x|p\rangle = C\,e^{+\frac{i}{\hbar}p\cdot x}$$

$$\hat{X}\,|x\rangle = x\,|x\rangle$$

$$\langle p|\hat{X}|x\rangle = x\,\langle p|x\rangle$$

$$\downarrow$$

$$= x\,\langle x|p\rangle^{*}$$

$$+i\hbar\frac{d}{dp}\underbrace{\langle p|x\rangle}_{C\,e^{-\frac{i}{\hbar}px}} = x\,C\,e^{-\frac{i}{\hbar}p\cdot x}$$

$$\therefore\quad \hat{P} \longrightarrow -i\hbar\frac{d}{dx}$$

$$\hat{x} \longrightarrow +i\hbar\frac{d}{dp}$$

↳ ZEIT ENTWICKLUNG EINES QUASI-KLASSISCHEN ZUSTANDES

- $$|\psi(t=0)\rangle = |\alpha_0\rangle$$

  ↳ $$\alpha_0 = \rho\, e^{i\phi}$$

  $$\rho, \phi \in \mathbb{R}$$

  $$\rho > 0$$

$$\Longrightarrow \quad |\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha_0)^n}{\sqrt{n!}} |n\rangle$$

- $$|\psi(t)\rangle$$

  $$\hat{H}\,|\psi(t)\rangle = i\hbar \frac{d}{dt} |\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha_0)^n}{\sqrt{n!}}\, e^{-\frac{i}{\hbar} E_n t} |n\rangle$$

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right)$$

$$|\psi(t)\rangle = e^{-i\omega t/2}\, e^{-|\alpha_0|^2/2} \cdot \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}}\, e^{-i\omega t \underline{\underline{n}}}$$

$$\left(\alpha_0\, e^{-i\omega t}\right)^n$$

$$\rightsquigarrow \quad \boxed{\alpha(t) \equiv \alpha_0\, e^{-i\omega t}} \quad \Longleftarrow$$

$$= \rho\, e^{-i(\omega t - \phi)} \quad \Longleftarrow$$

$$|\alpha(t)| = \rho = |\alpha_0|$$

$$\boxed{|\psi(t)\rangle = e^{-i\omega t/2}\, \underline{|\alpha(t)\rangle}}$$

$$\langle X \rangle_t = \sqrt{2}\, \mathrm{Re}\, \alpha(t)$$

$$= \underline{\sqrt{2}\, \rho}\, \cos(\omega t - \phi)$$

$$\underline{\langle x \rangle_t} = \underbrace{\sqrt{\frac{\hbar}{m\omega}}\, \sqrt{2}\, \rho}_{\underline{x_0}}\, \cos(\omega t - \phi)$$

$$\langle P \rangle_t = \sqrt{2}\ \mathrm{Im}\ \alpha(t)$$

$$= -\sqrt{2}\,\rho\,\sin(\omega t - \phi)$$

$$\langle p \rangle_t = -p_0 \sin(\omega t - \phi)$$

$$\langle p \rangle_t = m \frac{d}{dt} \langle x \rangle_t$$

LÖSUNGEN DES KLASSISCHEN OSZILLATORS !

$$\frac{\sigma_x}{|\langle x \rangle_t|} \simeq \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}\rho} = \frac{1}{2\rho} \ll 1$$

$\rho \gg 1$

$$\frac{\sigma_P}{|\langle P \rangle_t|} \simeq \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}\rho} = \frac{1}{2\rho} \ll 1$$

$$\rho = |\alpha|$$

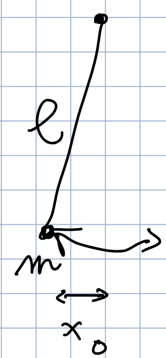
↳ **BEISPIEL**

PENDULUM

$\ell = 1\,m$

$m = 1\,g$

$x_0 = 10^{-6}\,m$



$$\omega = \sqrt{\frac{g}{\ell}} \simeq 3.\ s^{-1}$$

$$\alpha(t=0) = \alpha_0 = \rho = x_0 \sqrt{\frac{m\omega}{2\hbar}}$$

$$x_0 = \rho \sqrt{\frac{2\hbar}{m\omega}}$$

$$\rho \simeq 10^{-6} \underbrace{\left(\frac{10^{-3} \cdot 3}{10^{-34}}\right)^{1/2}}_{\sim 10^{15}}$$

$$\rho \simeq \ldots 10^{9}$$

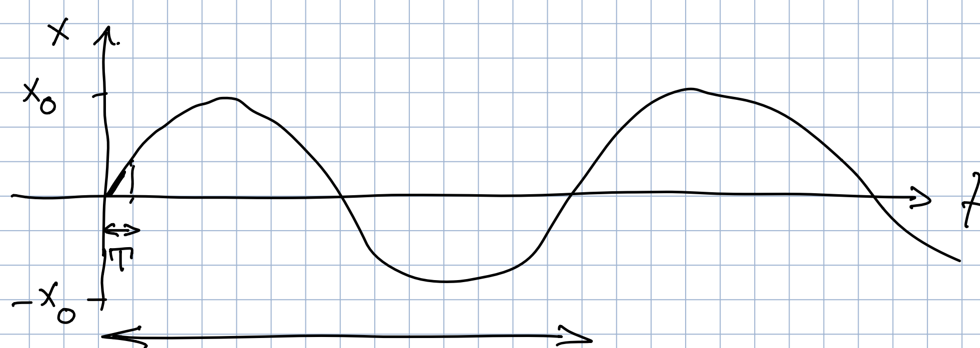
$$\frac{\sigma_x}{|\langle x \rangle|} \sim \frac{1}{\rho} \sim 10^{-10}$$

$$\frac{\sigma_p}{|\langle p \rangle|} \sim \frac{1}{\rho} \sim 10^{-10}$$

$\Longrightarrow$ "SCHRÖDINGER CAT" ZUSTAND

- $t = 0$

$$|\psi(t=0)\rangle = |\alpha\rangle$$



- $T \ll \frac{1}{\omega}$

$t \in [0, T]$ KOPPLUNG / STÖRUNG

$\rightsquigarrow \hat{W} = \hbar g \,(a^{+} a)^{2}$

$g \gg \omega$

$[0, T]$

EIGENZUSTÄNDE $\hat{W}$

$$\underbrace{\hat{H}}_{\hbar\omega\left(a^+a+\frac{1}{2}\right)} |n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$$

$$(a^+a)^2 |n\rangle = n^2 |n\rangle$$

$$\hat{W} |n\rangle = \hbar g n^2 |n\rangle$$

$$|\psi(t=0)\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\downarrow$$

$$\rightsquigarrow \quad |\psi(t=T)\rangle \simeq e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}\hbar g n^2 T} |n\rangle$$

SPEZIALFALL $T = \frac{\pi}{2g}$

$$e^{-ig n^2 T} = e^{-i\frac{\pi}{2}n^2} = \begin{cases} 1 , & n \text{ EVEN} \\ -i , & n \text{ ODD} \end{cases}$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}} + e^{+i\frac{\pi}{4}} (-1)^n \right)$$

$\nearrow$ n EVEN $\quad \frac{1}{\sqrt{2}}\left(2 \frac{1}{\sqrt{2}} \right) \stackrel{!}{=} 1$

$\searrow$ n ODD $\quad -\frac{1}{\sqrt{2}}\left(2i \frac{1}{\sqrt{2}} \right) \stackrel{!}{=} -i$

$$|\psi(t = T)\rangle = e^{-\frac{|\alpha|^2}{2}} \cdot \underbrace{\frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left\{ e^{-i\frac{\pi}{4}} + e^{i\frac{\pi}{4}} (-1)^n \right\} |n\rangle}$$

$$\frac{1}{\sqrt{2}} \left\{ e^{-i\frac{\pi}{4}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle + e^{+i\frac{\pi}{4}} \sum_n \frac{(-\alpha)^n}{\sqrt{n!}} |n\rangle \right\}$$

$$|\psi(t=T)\rangle = \frac{1}{\sqrt{2}}\left\{ e^{-i\frac{\pi}{4}}\,|\alpha\rangle + e^{+i\frac{\pi}{4}}\,|-\alpha\rangle \right\}$$