

\Rightarrow VORLESUNG 11 QM

\hookrightarrow 3.6 DIRAC'S BRA/KET NOTATION

$$\langle x | \Psi(t) \rangle \equiv \Psi(x, t)$$

$$\hat{X} |x\rangle = x |x\rangle$$

$$\langle p | \Psi(t) \rangle \equiv \Phi(p, t)$$

$$\hat{P} |p\rangle = p |p\rangle$$

$$\hat{H} |n\rangle = E_n |n\rangle$$

$$\langle n | \Psi(t) \rangle = c_n(t)$$

$$|n\rangle$$



$$|\Psi(t)\rangle = \sum_n c_n(t) |n\rangle$$

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→ DIRAC BRA/KET NOTATION

$$|\alpha\rangle = \sum_n a_n \underbrace{|n\rangle}_{\text{BASIS}}$$

1) DISKRET

ORTHONORM. $\langle n | m \rangle = \delta_{nm}$

VOLLSTÄNDIGKEIT $\sum_n \underbrace{|n\rangle}_{\text{KET}} \underbrace{\langle n|}_{\text{BRA}} = \mathbb{I}$ OPERATOR

$$\longrightarrow |\alpha\rangle = \sum_n a_n |n\rangle$$

$$|\beta\rangle = \sum_n b_n |n\rangle$$

$$\langle \beta | \alpha \rangle = \sum_n \sum_m b_m^* a_n \underbrace{\langle m | n \rangle}_{\delta_{nm}}$$

$$= \sum_n b_n^* a_n$$

$$= \begin{pmatrix} b_1^* & b_2^* & \dots \end{pmatrix}_{1 \times N} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

$$|\alpha\rangle \Leftrightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} \quad N \times 1$$

MATRIX $N \times 1$

$$\text{KET } |\beta\rangle \Leftrightarrow \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} \quad N \times 1$$

$$\text{BRA } \langle\beta| \Leftrightarrow \begin{pmatrix} b_1^* & b_2^* & \dots \end{pmatrix} \quad 1 \times N$$

$$\Rightarrow \underbrace{\langle\beta|}_{\text{BRA (C)}} \underbrace{|\alpha\rangle}_{\text{KET}} = \begin{matrix} & + \\ b & a \\ \uparrow & \uparrow \\ \text{MATRIX} & \text{ZEN} \end{matrix}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} \quad N \times 1$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix}$$

$N \times 1$

↑

$$b^\dagger = (b^T)^*$$

$$= \begin{pmatrix} b_1^* & b_2^* & \dots \end{pmatrix}$$

$1 \times N$

VOLLSTÄNDIGKEIT

$$\sum_n |n\rangle \langle n| = \mathbb{I}$$

$$\forall |\alpha\rangle$$

$$\left(\sum_n |n\rangle \langle n| \right) |\alpha\rangle = |\alpha\rangle$$

$$= \sum_n |n\rangle \underbrace{\langle n|\alpha\rangle}_{a_n}$$

$$= \sum_n a_n |n\rangle$$

$|\alpha\rangle$

2) KONTINUIERLICH

$\{|z\rangle\}$ BASIS

↳ DIRAC ORTHONORMALITÄT

$$\langle z' | z \rangle = \delta(z' - z)$$

↳ VOLLSTÄNDIGKEIT

$$\int dz |z\rangle \langle z| = \mathbb{I}$$

$$\forall |\alpha\rangle$$

$$|\alpha\rangle = \int dz |z\rangle \underbrace{\langle z | \alpha \rangle}_{\alpha(z)}$$

$$|\alpha\rangle = \int dz \alpha(z) |z\rangle$$

$$\forall |z'\rangle$$

$$\underbrace{\langle z' | \alpha \rangle}_{\alpha(z')} = \int dz \alpha(z) \underbrace{\langle z' | z \rangle}_{\delta(z' - z)}$$

$$\begin{array}{ccc} \uparrow & & \\ \alpha(z') & \stackrel{\nabla}{=} & \alpha(z') \end{array}$$

$$\delta(z-z')$$

↳ OPERATOR

$$|\beta\rangle = \hat{Q}|\alpha\rangle$$

$$|\alpha\rangle = \sum_n a_n |n\rangle$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} \quad N \times 1$$

$$|\beta\rangle = \sum_n b_n |n\rangle$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix}$$

$$\hat{Q}|\alpha\rangle = \sum_n a_n \hat{Q}|n\rangle$$

$$\sum_n |n\rangle \langle n| = \mathbb{I}$$

$$= \sum_n \sum_m a_m |m\rangle \underbrace{\langle m | \hat{Q} | n \rangle}_{Q_{mn}}$$

$$\langle m | \hat{Q} | n \rangle$$

$$| \hat{Q} n \rangle$$

$$\hat{Q}^\dagger = \hat{Q}$$

$$= \langle \hat{Q} m | n \rangle$$

$$= \langle n | \hat{Q} m \rangle^*$$

$$= \langle n | \hat{Q} | m \rangle^*$$

$$\underbrace{\langle m | \hat{Q} | n \rangle}_{Q_{mn}} = \underbrace{\langle n | \hat{Q} | m \rangle^*}_{Q_{nm}^*}$$

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & \dots \\ Q_{21} & & \\ \vdots & & \end{pmatrix}$$

$$N \times N$$

$$\begin{aligned}
 (Q)_{mn} &= (Q^{T*})_{nm} \\
 &= Q_{nm}^*
 \end{aligned}$$

$$Q = Q^\dagger = (Q^T)^*$$

$$\begin{aligned}
 |\beta\rangle &= \hat{Q} |\alpha\rangle = \sum_n \sum_m a_m Q_{mn} |m\rangle \\
 &= \sum_m \left(\underbrace{\sum_n Q_{mn} a_n}_{b_m} \right) |m\rangle
 \end{aligned}$$

$$\begin{aligned}
 b_m &= \sum_n Q_{mn} a_n \\
 \text{Venn} \downarrow & \\
 \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} &= \begin{pmatrix} Q_{11} & Q_{12} & \dots \\ Q_{21} & & \\ \vdots & & \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}
 \end{aligned}$$

$\mathbb{N} \times 1$ $\mathbb{N} \times \mathbb{N}$ $\mathbb{N} \times 1$ b $=$ \hat{Q} a \updownarrow $|\beta\rangle$ $=$ \hat{Q} $|\alpha\rangle$

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\Rightarrow SCHRÖDINGER'S KATZE / QUASI-KLASSISCHE ZUSTÄNDE

\hookrightarrow 1) QUASI-KLASSISCHE
 ZUSTÄNDE DES H.O.

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\langle x | \hat{H} | \psi \rangle = E \langle x | \psi \rangle$$

$$\langle x | \frac{\hat{P}^2}{2m} | \psi \rangle + \langle x | \frac{1}{2} m \omega^2 \hat{x}^2 | \psi \rangle$$

$$= E \psi(x)$$

$$= -i\hbar \frac{d}{dx} \frac{d}{dx} \underbrace{\psi(x)}_{\langle x | \psi \rangle} + \underbrace{\langle x | \psi \rangle}_{\psi(x)} \frac{1}{2} m \omega^2 x^2$$

$$\leadsto \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 \right) \psi(x) = E \psi(x)$$

$$\hat{H}|n\rangle = E_n |n\rangle$$

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \quad n = 0, 1, \dots$$

$$\hat{H} = \hbar\omega \left(\overset{\uparrow}{a^+} a + \frac{1}{2} \right)$$

$$\begin{aligned} a &= a_- \\ a^+ &= a_+ \end{aligned}$$

$$a |n\rangle = \sqrt{n} |n-1\rangle \quad \Leftarrow$$

$$a^+ |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$a^+ a |n\rangle = \underline{n} |n\rangle$$

$$[a, a^+] = 1$$

$$\begin{aligned} \hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \\ &= \frac{\hbar\omega}{2} \left(\frac{\hat{p}^2}{\hbar\omega m} + \frac{m\omega}{\hbar} \hat{x}^2 \right) \end{aligned}$$

$$\begin{cases} \hat{X} \equiv \sqrt{\frac{m\omega}{\hbar}} \hat{x} \\ \hat{P} \equiv \sqrt{\frac{1}{\hbar\omega m}} \hat{p} \end{cases}$$

$$\begin{aligned} \Rightarrow \hat{H} &= \frac{\hbar\omega}{2} \left(\hat{X}^2 + \hat{P}^2 \right) \\ &= \frac{\hbar\omega}{2} \left(\hat{X}^2 - (i\hat{P})^2 \right) \\ &= \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) \end{aligned}$$

$$\begin{cases} a = \frac{1}{\sqrt{2}} \left(\hat{X} + i\hat{P} \right) \\ a^\dagger = \frac{1}{\sqrt{2}} \left(\hat{X} - i\hat{P} \right) \end{cases}$$

$$a |n=0\rangle = 0$$

$$(\hat{X} + i\hat{P}) |n=0\rangle = 0$$



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$$\langle x | \hat{X} + i \hat{P} | n=0 \rangle = 0$$

$$\left(\cancel{X} + \frac{d}{dX} \right) \underbrace{\langle x | n=0 \rangle}_{\psi_0(x)} = 0$$

QUASI-KLASSISCHER ZUSTAND

$$\leadsto || \alpha |\alpha\rangle \equiv \alpha |\alpha\rangle \quad \alpha \in \mathbb{C}$$

$$a |n\rangle = \sqrt{n} |n-1\rangle$$

$$\langle \alpha | a^\dagger = \langle \alpha | \alpha^*$$

$$\begin{aligned} \langle \alpha | a^\dagger | \beta \rangle &= \langle \alpha | a^\dagger \beta \rangle \\ &= \langle a^\dagger \beta | \alpha \rangle^* \\ &\downarrow \\ \langle \beta | a | \alpha \rangle^* \end{aligned}$$

$$\begin{aligned}\langle \alpha | a^\dagger | \alpha \rangle &= \langle \alpha | a | \alpha \rangle^* \\ &= \alpha^* \underbrace{\langle \alpha | \alpha \rangle}_1\end{aligned}$$

$$\hat{H} |n\rangle = \hbar \omega \left(n + \frac{1}{2}\right) |n\rangle$$

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha_n |n\rangle$$

$$\alpha_n = \langle n | \alpha \rangle$$

$$a |\alpha\rangle = \alpha |\alpha\rangle$$

$$a |\alpha\rangle = \sum_{n=1}^{\infty} \alpha_n a |n\rangle$$

$$= \sum_{n=1}^{\infty} \alpha_n \sqrt{n} |n-1\rangle$$

$$= \sum_{n=0}^{\infty} \alpha_{n+1} \sqrt{n+1} |n\rangle$$

$n \rightarrow n+1$

$$= \alpha \sum_{n=0}^{\infty} \alpha_n |n\rangle$$

$$\alpha_{n+1} \sqrt{n+1} = \alpha \alpha_n$$

$$\alpha_{n+1} = \frac{\alpha}{\sqrt{n+1}} \alpha_n$$

$$\alpha_0 = C$$

$$\alpha_1 = C \frac{\alpha}{\sqrt{1}}$$

$$\alpha_2 = C \frac{\alpha^2}{\sqrt{1 \cdot 2}}$$

...

$$\alpha_n = C \frac{\alpha^n}{\sqrt{n!}}$$

$$|\alpha\rangle = C \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$1 = \langle \alpha | \alpha \rangle$$

$$1 = |C|^2 \sum_n \sum_m \frac{(\alpha^*)^n}{\sqrt{n!}} \frac{\alpha^m}{\sqrt{m!}} \underbrace{\langle n | m \rangle}_{\delta_{nm}}$$

$$= |C|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}$$

$$e^{|\alpha|^2}$$

$$|C| = e^{-|\alpha|^2/2}$$



$$\Rightarrow |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$a |\alpha\rangle = \alpha |\alpha\rangle$$

$$\langle \alpha | \alpha \rangle = 1$$

KOHÄRENTER ZUSTAND

$$\hookrightarrow \langle \hat{H} \rangle, \quad \langle \hat{x} \rangle, \quad \langle \hat{p} \rangle$$

$$\begin{aligned} \bullet \quad \langle \hat{H} \rangle &= \langle \alpha | \hat{H} | \alpha \rangle \\ &= \hbar \omega \langle \alpha | \underbrace{a^\dagger a + \frac{1}{2}} | \alpha \rangle \\ &= \hbar \omega \langle \alpha | \underbrace{a^\dagger}_{\alpha^*} \underbrace{a}_{\langle \alpha | \alpha \rangle = 1} | \alpha \rangle + \frac{\hbar \omega}{2} \end{aligned}$$

$$\langle \hat{H} \rangle = \hbar \omega \left(|\alpha|^2 + \frac{1}{2} \right)$$

$|\alpha|^2$ MITTLERE ANREGUNGS-
NIVEAU DES H. O.

$$\bullet \quad \langle \hat{x} \rangle \quad \left\{ \begin{array}{l} \hat{x} \equiv \sqrt{\frac{m\omega}{\hbar}} \hat{x} \\ \hat{p} \equiv \sqrt{\frac{\hbar}{m\omega}} \hat{p} \end{array} \right.$$

$$\langle \hat{X} \rangle = \sqrt{\frac{\hbar}{m\omega}} \langle \hat{X} \rangle$$

$$\begin{cases} a = \frac{1}{\sqrt{2}} (\hat{X} + i \hat{P}) \\ a^\dagger = \frac{1}{\sqrt{2}} (\hat{X} - i \hat{P}) \end{cases}$$

$$\begin{aligned} \hat{X} &= \frac{1}{\sqrt{2}} (a + a^\dagger) \\ i \hat{P} &= \frac{1}{\sqrt{2}} (a - a^\dagger) \end{aligned}$$

$$\begin{aligned} \langle \hat{X} \rangle &= \langle \alpha | \frac{1}{\sqrt{2}} (a + a^\dagger) | \alpha \rangle \\ &= \frac{1}{\sqrt{2}} (\alpha + \alpha^*) \underbrace{\langle \alpha | \alpha \rangle}_1 \\ &= \sqrt{2} \operatorname{Re} \alpha \end{aligned}$$

$$\langle \hat{X} \rangle = \sqrt{\frac{\hbar}{m\omega}} \sqrt{2} \operatorname{Re} \alpha$$

$$\begin{aligned}
 \langle \hat{P} \rangle &= -\frac{i}{\sqrt{2}} \langle \alpha | a - a^\dagger | \alpha \rangle \\
 &= -\frac{i}{\sqrt{2}} (\alpha - \alpha^*) \\
 &= \sqrt{2} \operatorname{Im} \alpha
 \end{aligned}$$

$$\langle \hat{P} \rangle = \sqrt{\hbar \omega m} \langle \hat{P} \rangle$$

$$\langle \hat{P} \rangle = \sqrt{\hbar \omega m} \sqrt{2} \operatorname{Im} \alpha.$$

$$a_x^2, a_p^2$$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$