⟹ VORLESUNG 10 QM

↳ 3.4 STATISTISCHE INTERPRETATION

$$Q = Q^{\dagger}$$

↳ EIGENWERTE REELE ZAHLEN

- DISKRETES SPECTRUM

$$Q\,|f_n\rangle = q_n\,|f_n\rangle$$

$$|\psi\rangle = \sum_n c_n\,|f_n\rangle$$

$$c_n = \langle f_n|\psi\rangle$$

$|c_n|^2$ ⤳ WAHRSCHEINLICHKEIT

- KONTINUIERLICHES SPEKTRUM

$$Q\,|f_z\rangle = q(z)\,|f_z\rangle$$

z KONTINUIERLICH

$$|\psi\rangle = \int dz\; c(z)\,|f_z\rangle$$

$|c(z)|^2 dz$ $\quad z,\ z+dz$

- $$\sum_n |c_n|^2 = 1$$

$$|\psi\rangle = \sum_n c_n |f_n\rangle$$

$$\langle\psi|\psi\rangle = \sum_n \sum_{n'} c_n c_{n'}^* \underbrace{\langle f_{n'}|f_n\rangle}_{\delta_{nn'}}$$

$$= \sum_n |c_n|^2$$

$$= \underline{1}$$

- $\langle Q \rangle$ IM ZUSTAND $|\psi\rangle$

$$= \langle \psi | \hat{Q} \psi \rangle$$

$$= \sum_n \sum_{n'} c_n c_{n'}^* \langle f_{n'} | \underbrace{\hat{Q} f_n}_{q_n |f_n\rangle} \rangle$$

$$\delta_{nn'}$$

$$\langle Q \rangle = \sum_n q_n |c_n|^2$$

- WELLENFUNKTION IM IMPULSRAUM

$$\hat{p}\,|f_p\rangle = p\,|f_p\rangle$$

$$\hat{p}\, f_p(x) = p\, f_p(x)$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar} p \cdot x}$$

$$\langle f_p | f_{p'} \rangle = \int dx\, f_p^*(x)\, f_{p'}(x)$$
$$= \delta(p - p')$$

DIRAC ORTHONORMALITÄT

$$|\psi\rangle = \int dp\, c(p)\, |f_p\rangle$$

$$c(p) = \langle \underline{\underline{f_p}} | \psi \rangle$$

$$c(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-\frac{i}{\hbar} p x}\, \psi(x)$$

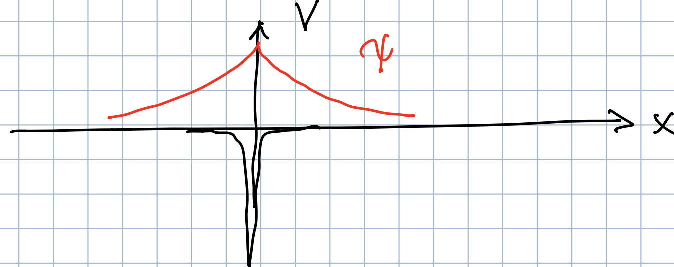
$$\rightsquigarrow \quad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-\frac{i}{\hbar} p x}\, \Psi(x,t)$$

W.F. IM IMPULSRAUM

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \; e^{\frac{i}{\hbar} p x} \; \Phi(p,t)$$

$$[p, p+dp] \longrightarrow |\Phi(p,t)|^2 \, dp$$

BEISPIEL



$$V(x) = -\alpha\,\delta(x) \qquad (\alpha > 0)$$

$$E = -\frac{\hbar^2}{2m} K^2 \qquad K = \frac{m\alpha}{\hbar^2}$$

$$\psi(x) = \sqrt{K}\; e^{-K|x|}$$

$$\Psi(x,t) = \sqrt{K}\; e^{-K|x|}\; e^{-\frac{i}{\hbar} E t}$$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx \; e^{-\frac{i}{\hbar} p x} \sqrt{K}\, e^{-K|x|}\, e^{-\frac{i}{\hbar} E t}$$

$$= \frac{\sqrt{K}}{\sqrt{2\pi\hbar}}\, e^{-\frac{i}{\hbar} E t} \left\{ \int_{-\infty}^{0} dx \; e^{\left(K - \frac{i}{\hbar} p\right) x} \right.$$

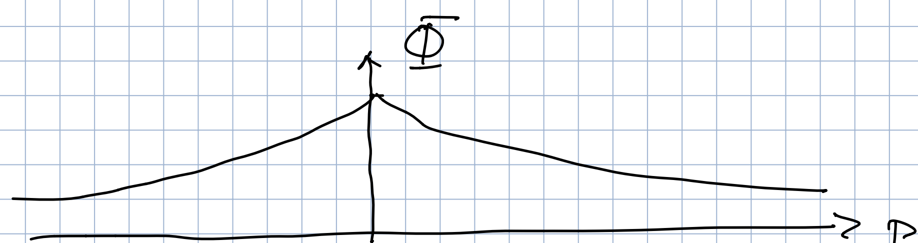
$$+ \int_0^\infty dx \quad e^{-(K+\frac{i}{\hbar}p)x} \Big\}$$

$$= \frac{\sqrt{K}}{\sqrt{2\pi\hbar}} \, e^{-\frac{i}{\hbar}Et} \Big\{ \frac{1}{K-\frac{i}{\hbar}p} \, e^{(K-\frac{i}{\hbar}p)x}\Big|_{\cancel{-\infty}}^{0}$$

$$- \frac{1}{(K+\frac{i}{\hbar}p)} \, e^{-(K+\frac{i}{\hbar}p)}\Big|_{0}^{\cancel{+\infty}} \Big\}$$

$$= \frac{\sqrt{K}}{\sqrt{2\pi\hbar}} \, e^{-\frac{i}{\hbar}Et} \Big\{ \underbrace{\frac{1}{K-\frac{i}{\hbar}p} + \frac{1}{K+\frac{i}{\hbar}p}}_{\dfrac{2K}{K^2+\frac{p^2}{\hbar^2}}} \Big\}$$

$$\Phi(p,t) = \sqrt{\frac{2}{\pi}} \, \frac{(\hbar K)^{3/2}}{(\hbar K)^2 + p^2} \quad e^{-\frac{i}{\hbar}Et}$$



$$P(p \geq \hbar K)$$

$$= \int_{\hbar K}^{\infty} dp \; |\Phi(p,t)|^2$$

$$\simeq 0.09$$

↳ 3.5 UNSCHÄRFE PRINZIP

$$\hat{A}, \quad \hat{B} \qquad (\hat{A} = \hat{A}^{+}, \quad \hat{B} = \hat{B}^{+})$$

$$\langle \hat{A} \rangle = \langle \Psi | \hat{A} \Psi \rangle$$

$$\langle \hat{B} \rangle = \langle \Psi | \hat{B} \Psi \rangle$$

$$\sigma_A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle$$

$$= \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)^2 \Psi \rangle$$

$\hat{A} - \langle \hat{A} \rangle$ HERMITISCH

$$= \langle \underbrace{(\hat{A} - \langle \hat{A} \rangle) \Psi}_{f} \,|\, \underbrace{(\hat{A} - \langle \hat{A} \rangle) \Psi}_{f} \rangle$$

$$= \langle f | f \rangle$$

$$\sigma_B^2 = \langle (\hat{B} - \langle \hat{B} \rangle)^2 \rangle$$

$$= \langle \underbrace{(\hat{B} - \langle \hat{B} \rangle) \Psi}_{g} \,|\, \underbrace{(\hat{B} - \langle \hat{B} \rangle) \Psi}_{g} \rangle$$

$$= \langle g | g \rangle$$

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \cdot \langle g | g \rangle$$

$$\Big\downarrow \text{ SCHWARZ UNGL.}$$

$$\geq \left| \langle f | g \rangle \right|^2$$

$$|z|^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2$$

$$\geq (\mathrm{Im}\, z)^2 = \left(\frac{1}{2i} \left(z - z^* \right) \right)^2 .$$

$$z = \langle f | g \rangle$$

$$= \langle (\hat{A} - \langle A \rangle) \Psi \,|\, (\hat{B} - \langle B \rangle) \Psi \rangle$$

$$= \langle \Psi | \underbrace{(\hat{A} - \langle \hat{A} \rangle)(\hat{B} - \langle \hat{B} \rangle)} \Psi \rangle$$

$$= \langle \Psi | \left(\hat{A}\hat{B} - \langle A \rangle \hat{B} - \langle B \rangle \hat{A} + \langle A \rangle \langle B \rangle \right) \Psi \rangle$$

$$= \langle \Psi | \hat{A}\hat{B} \Psi \rangle - \langle A \rangle \underbrace{\langle \Psi | \hat{B} \Psi \rangle}_{\langle B \rangle}$$

$$z^* = \langle f | g \rangle^*$$

$$= \langle g | f \rangle$$

$$= \langle \Psi | \hat{B}\hat{A} \Psi \rangle - \langle A \rangle \langle B \rangle$$

$$z - z^* = \langle \Psi | \underbrace{\left(\hat{A}\hat{B} - \hat{B}\hat{A} \right)}_{[\hat{A}, \hat{B}]} \Psi \rangle$$

KOMMUTATOR.

$$\therefore \quad \boxed{\sigma_A^2 \, \sigma_B^2 \geq \underbrace{\left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}_{\text{POSITIV !}}}$$

$$\hat{A}, \quad \hat{B} \qquad \hat{A}^\dagger = \hat{A}, \quad \hat{B}^\dagger = \hat{B}$$

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$(\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger = \hat{B}^\dagger \hat{A}^\dagger - \hat{A}^\dagger \hat{B}^\dagger$$

$$A^\dagger = (A^T)^*$$

$$= \hat{B}\hat{A} - \hat{A}\hat{B}$$

$$[\hat{A}, \hat{B}]^{\dagger} = -[\hat{A}, \hat{B}]$$

ANTI-HERMITISCH

REIN IMAG EIGENWERTE

↳ $\sigma_x \; \sigma_p$ POSITION – IMPULS UNSCHÄRFE

$$\hat{A} = \hat{x}$$

$$\hat{B} = \hat{p}$$

$$\sigma_x^2 \, \sigma_p^2 \geq \left(\frac{1}{2i} \langle [\hat{x}, \hat{p}] \rangle \right)^2$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\sigma_x^2 \, \sigma_p^2 \geq \left(\frac{1}{2i} \, i\hbar \right)^2$$

$$\sigma_x \, \sigma_p \geq \frac{\hbar}{2}$$

$$\Delta x \cdot \Delta p$$

$[\hat{A}, \hat{B}] \neq 0 \iff$ A, B SIND INKOMPATIBLE OBSERVABLEN

$\Updownarrow$

KEINE GEMEINSAME EIGENFUNKTIONEN

$[\hat{A}, \hat{B}] = 0$ KOMPATIBLE OBSERVABLEN

GEMEINSAME EIGENFUNKTIONEN

↳ MINIMALE UNSCHÄRFE

$$\sigma_A^2 \sigma_B^2 = \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

$$\sigma_x \sigma_p = \frac{\hbar}{2} \quad ?$$

1) $\langle f | f \rangle \langle g | g \rangle = | \langle f | g \rangle |^2$

$$\Updownarrow$$

$$g(x) = c \, f(x)$$

2) $$|z|^2 = |\langle f|g\rangle|^2 = \left(\operatorname{Im}\langle f|g\rangle\right)^2$$

$$\Updownarrow$$

$$|c|^2 |\langle f|f\rangle|^2 = |\langle f|f\rangle|^2 \left(\operatorname{Im} c\right)^2$$

$$c = \underline{\underline{ia}} \qquad (a \in \mathbb{R})$$

$$\therefore \quad \underline{g(x) = ia\, f(x)} \qquad (a \in \mathbb{R})$$

$$f(x) \Rightarrow |(\hat{x} - \langle x\rangle)\Psi\rangle$$

$$g(x) \Rightarrow |(\hat{p} - \langle p\rangle)\Psi\rangle$$

$$\rightsquigarrow \left(-i\hbar \frac{d}{dx} - \langle p\rangle\right)\Psi(x) = ia\left(x - \langle x\rangle\right)\Psi(x)$$

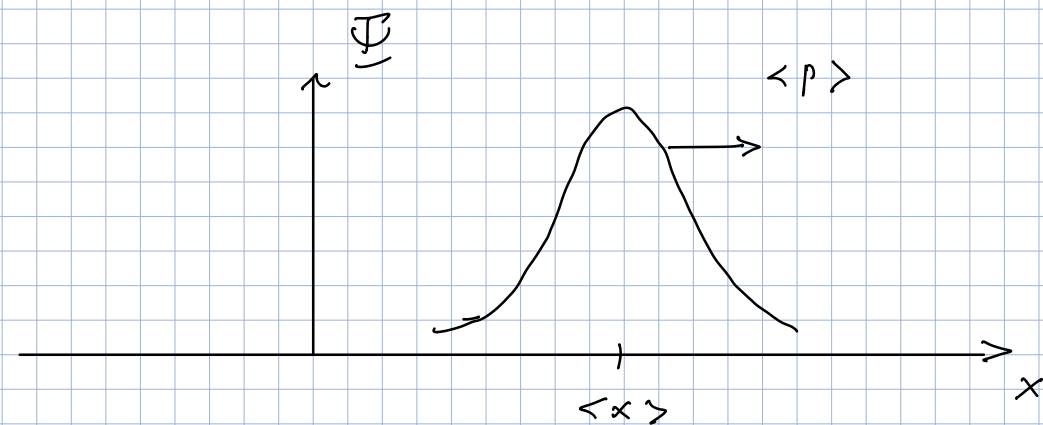
$$\rightsquigarrow \quad \Psi(x) = e^{-\frac{i}{\hbar}\langle p\rangle x}\, \Phi(x)$$

$$\left(-i\hbar\frac{d}{dx} - \langle p\rangle\right)\Psi = e^{-\frac{i}{\hbar}\langle p\rangle x}\left(-i\hbar\frac{d}{dx}\Phi\right)$$

$$-\cancel{i}\hbar \frac{d}{dx} \Phi(x) = \cancel{i} a \left(x - \langle x \rangle\right) \Phi(x)$$

$$\Phi(x) = C\, e^{-\frac{a}{2\hbar}\left(x - \langle x \rangle\right)^2}$$

$$\Psi(x) = C\, e^{-\frac{i}{\hbar}\langle p \rangle x}\, e^{-\frac{a}{2\hbar}\left(x - \langle x \rangle\right)^2}$$



MINIMALER UNSCHÄRFE

- ENERGIE – ZEIT UNSCHÄRFE

$\rightsquigarrow \quad \Delta E \; \underline{\underline{\Delta t}} \geq \frac{\hbar}{2}$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2.$$

$E \hookrightarrow \hat{H}$

$t \quad ?$

$$\frac{d}{dt} \langle \hat{Q} \rangle \qquad \hat{Q}^+ = \hat{Q}$$

$$= \frac{d}{dt} \langle \Psi | \hat{Q} \Psi \rangle$$

$$= \langle \underline{\underline{\frac{\partial \Psi}{\partial t}}} | \hat{Q} \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} \Psi \rangle + \langle \Psi | \hat{Q} \underline{\underline{\frac{\partial \Psi}{\partial t}}} \rangle$$

$$| \hat{H} \Psi \rangle = i\hbar \left| \frac{\partial \Psi}{\partial t} \right\rangle$$

$$\left| \frac{\partial \Psi}{\partial t} \right\rangle = -\frac{i}{\hbar} | \hat{H} \Psi \rangle$$

$$= \frac{i}{\hbar} \langle \hat{H} \Psi | \hat{Q} \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} \Psi \rangle - \frac{i}{\hbar} \langle \Psi | \hat{Q} \hat{H} \Psi \rangle$$

$$= \frac{i}{\hbar} \langle \Psi | (\hat{H}\hat{Q} - \hat{Q}\hat{H}) \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} \Psi \rangle$$

$$\frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle$$

t- ÄNDERUNG VON Ψ

t- ÄNDERUNG VON $\hat{Q}$

$$\sigma_H^2 \, \sigma_Q^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2$$

$$\Downarrow \qquad \langle \frac{\partial \hat{Q}}{\partial t} \rangle = 0$$

$$\sigma_H \, \sigma_Q \geq \left(\frac{\hbar}{2} \right) \left| \frac{d}{dt} \langle Q \rangle \right|$$

$$\sigma_H = (\Delta E)$$

$$\Delta t = \frac{\sigma_Q}{\left| \frac{d}{dt} \langle Q \rangle \right|}$$

$$\sigma_Q \approx \underbrace{\left| \frac{d}{dt} \langle Q \rangle \right| \cdot \underbrace{\Delta t}_{}}_{\substack{\text{ÄNDERUNG } \langle Q \rangle \\ \text{ÜBER } \Delta t}}$$

$$\Delta E \; \Delta t \geq \frac{\hbar}{2}$$