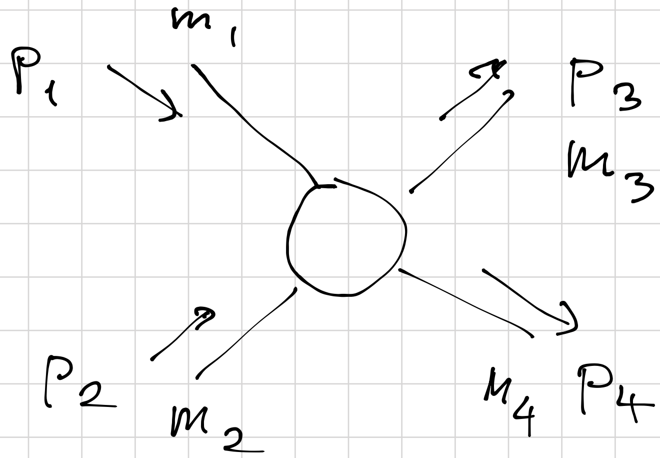
Lecture 13 14.12.2020

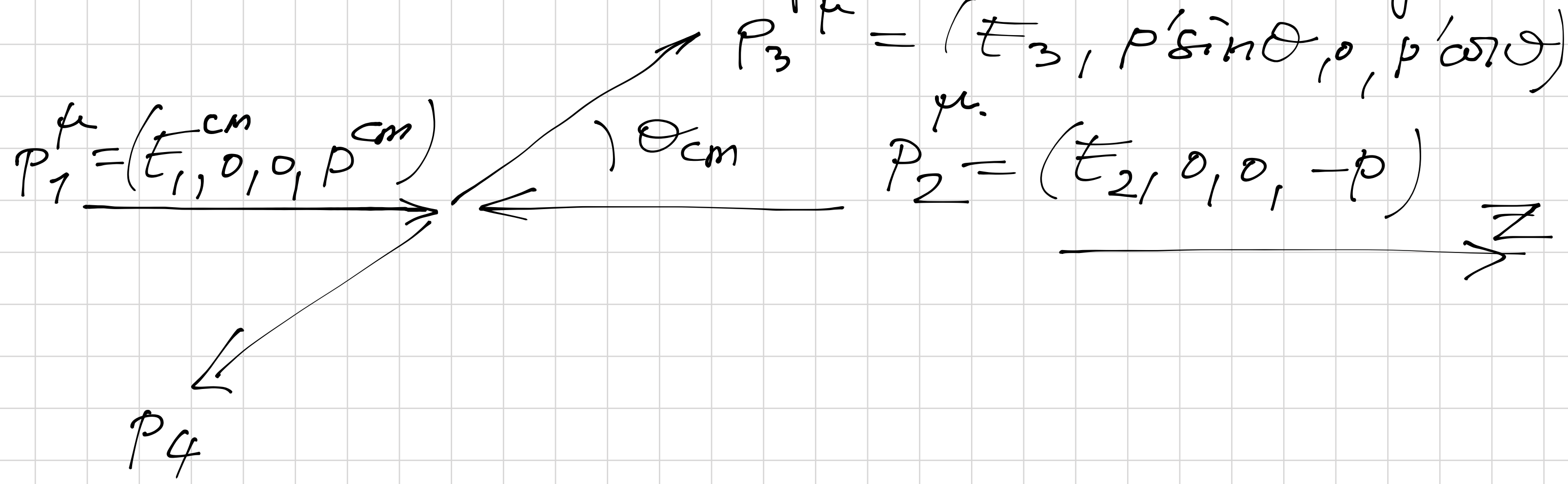
For the exercises:



1. Center of momentum

$$\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_3 + \vec{p}_4$$

Advantage is: energies and momenta independent of angles

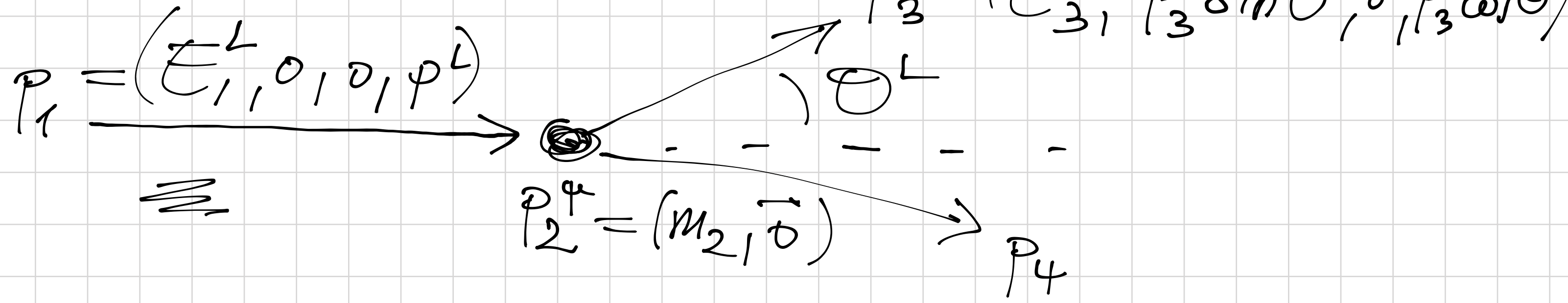


Natural frame for colliders

2. Laboratory frame

Defined: $\vec{p}_2 = 0$ $\quad p_2^\mu = (m_2, \vec{0})$

Natural for fixed target exps.



Initial energy indep. of θ
But final energy $E_3^L = E_3^L(\theta^L)$
recall Compton effect

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$p_4^2 = m_4^2 = (p_1 + p_2 - p_3)^2 = m_2^2 + t + 2m_2(E_1^L - E_3^L)$$

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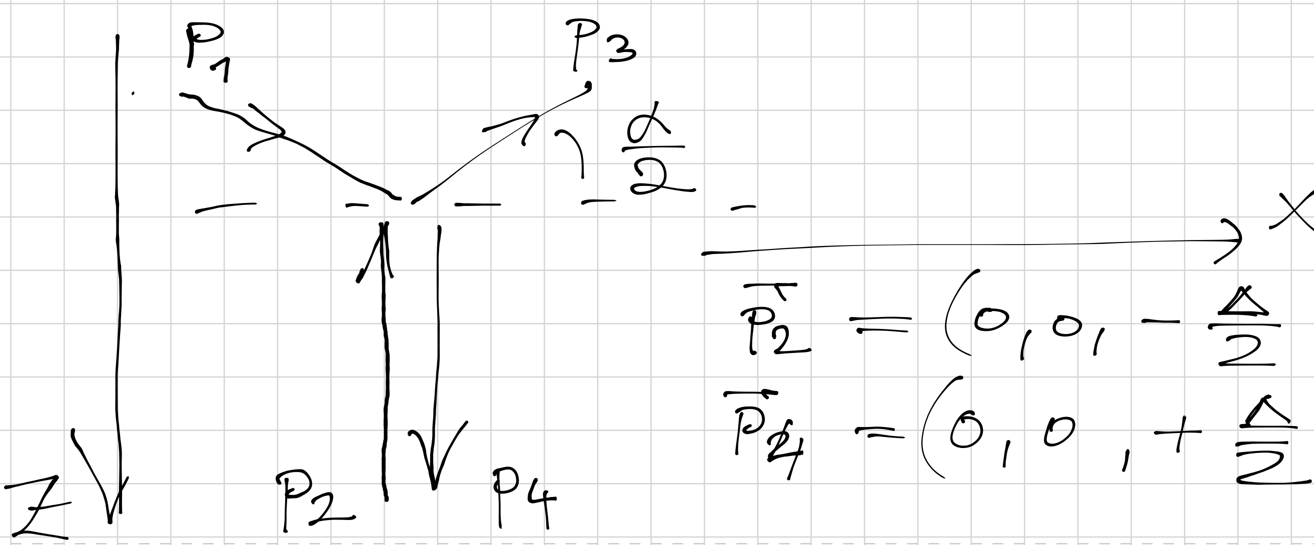
3. Breit (brick-wall) frame
Defined by $\vec{p}_2 + \vec{p}_4 = 0$
Useful if you scatter off a heavy target (e.g. electron-nucleus)
Static limit (no recoil) natural
Advantage: $m_2 = m_4$, $E_2 = E_4$

$$P^\mu = \frac{1}{2}(p_2 + p_4)^\mu = (P^0, \vec{0})$$
$$\Delta^\mu = (p_4 - p_2)^\mu \qquad \Delta^2 = t$$
$$(\Delta \cdot P) = \frac{1}{2}(p_4^2 - p_2^2) \underset{m_2 = m_4}{=} \frac{1}{2}(m_4^2 - m_2^2) = 0$$

$\Delta \to$ only spatial $\qquad \vec{\Delta} \parallel \hat{e}_z$



$$\vec{p}_2 = (0, 0, -\frac{\Delta}{2})$$
$$\vec{p}_4 = (0, 0, +\frac{\Delta}{2})$$

Back to spin-1 field

Quantized spin-1 field last week

$$A^\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_k \left[\hat{a}_k \varepsilon_k^\mu(p) \underline{e^{-ipx}} + \hat{a}_k^+ \varepsilon_k^{*\mu}(p) \underline{e^{ipx}} \right]$$

Massive spin-1

E.o.M. $\begin{cases} (\Box + m^2) A^\mu = 0 \\ m^2 \underbrace{\partial_\mu A^\mu}_{} = 0 \end{cases}$ → Fourier transform

→ removes the spin-0 component

restricts the ε_μ

$$p_\mu \varepsilon^\mu = 0$$

3 independent polarizations

$p^\mu = (\omega, 0, 0, p)$ $\qquad \varepsilon^\mu_\pm(p) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$

$$\varepsilon^\mu_L(p) = \left(\frac{p}{m}, 0, 0, \frac{\omega}{m}\right)$$

Vectors $\varepsilon_k(p)$ form a little Lorentz group

Transformations $\Lambda^{\mu\nu}$: $\Lambda^{\mu\nu} p_\nu = p^\mu$

SO(3) 3D rotations

Massless spin-1

$$\begin{cases} (\Box + m^2) A^\mu = 0 \\ \underline{m^2 \partial_\mu A^\mu = 0} \end{cases} \xrightarrow{m \to 0} \begin{cases} \Box A^\mu = 0 \\ \text{Gauge inv.} \end{cases}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$ invariant under $A^\mu \to A^\mu + \partial^\mu \alpha$

arbitrary α

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Invariance: physics the same for <u>any</u> α

$\vec{\nabla}\vec{A} = 0$ Coulomb gauge

Additionally $A^0 = 0$ (comes out of E.o.M.)

For $p^\mu = (\omega, 0, 0, \omega)$

$$\varepsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) \quad \text{only 2 d.o.f.}$$

In Lorentz gauge: $\varepsilon^\mu_f = (1, 0, 0, 1) \parallel p^\mu$

$$\varepsilon_f^2 = 0$$

There seems to be 3 d.o.f. ?

If we start with just $\varepsilon^\mu_\pm$ the little Lorentz group will mix them with ε^μ_f

<u>Specific example</u> $\quad p^\mu = (\omega, 0, 0, \omega)$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \frac{3}{2} & 1 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \end{pmatrix}$$

1. Check that Λ is a L.T. ✓

$$g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

" 1

$$g_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \begin{pmatrix} \left(\frac{3}{2}\right)^2 - 1 - \left(-\frac{1}{2}\right)^2 & \frac{3}{2} - 1 - \frac{1}{2} \xrightarrow{} 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= g_{\alpha\beta}$$

2. Check $\Lambda p = p$ ✓

$$\Lambda^{\mu}{}_{\nu} \cdot p^{\nu} = \omega \begin{pmatrix} 3/2 & 1 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \omega \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = p^{\mu}$$

Let's act $\omega \cdot \Lambda$ upon $\varepsilon_{\pm}$

$$\Lambda^{\mu}{}_{\nu} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ \pm i \\ 1 \end{pmatrix} = \varepsilon^{\mu}_{\pm} + \frac{1}{\sqrt{2}\omega} p^{\mu}$$

$$= \varepsilon^{\mu}_{\pm} + \frac{1}{\sqrt{2}} \underline{\underline{\varepsilon^{\mu}_{f}}}$$

Physical amplitude

$$\sim \varepsilon_{\pm\mu} M^{\mu} \xrightarrow{\Lambda} \left(\varepsilon_{\pm\mu} + \frac{1}{\sqrt{2}\omega} p_{\mu}\right) M'^{\mu}$$

$$M^{\mu} \rightarrow \Lambda^{\mu\nu} M_{\nu} = M'^{\mu}$$

In order for the polarizations to be physical

$$\boxed{p_\mu M^\mu = 0}$$ <u>Ward ID</u>

Follows from Lorentz invariance
Closely related to G.I.

$$A^\mu \to A^\mu + \partial^\mu \alpha$$

But in momentum space $\partial^\mu \sim p^\mu$

$$\varepsilon^\mu \to \varepsilon^\mu + p^\mu \cdot \tilde{\alpha}$$ should also be a symmetry

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Next step: photon propagator

$$\langle 0 | T\{A^\mu(x) A^\nu(y)\} | 0 \rangle = i \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \Pi^{\mu\nu}(p)$$

We solved for a Green's function

G.F.
$$(\Box + m^2)\varphi = 0$$
$$(\Box_x + m^2) G(x, y) = \delta^4(y)$$

Start w. $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A^\mu J_\mu$

E.o.M. $$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\hookrightarrow \partial_\mu F^{\mu\nu} = J^\nu$$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = J^\nu$$

In momentum space $\partial^\mu \to i p^\mu$

$$\underline{\left(-p^2 g^{\mu\nu} + p^\mu p^\nu\right)} A_\mu = J^\nu$$

$$A_\mu = \Pi_{\mu\nu} J^\nu$$

$$\left(-g^{\mu\nu} p^2 + p^\mu p^\nu\right) \Pi_\mu{}^\alpha = g^{\nu\alpha}$$

$$\underline{\det\left(-g^{\mu\nu} p^2 + p^\mu p^\nu\right)} = 0$$

p^μ is eigenvector, eigenvalue $= \emptyset$

$$\left(-g^{\mu\nu} p^2 + p^\mu p^\nu\right) p_\nu = \underbrace{\left(-p^2 + p^2\right)}_{=0} p^\mu$$

This is a math. manifestation of G.I.:
A^μ is not uniquely determined by J^ν
(there is always freedom
$A^\mu \to A^\mu + \partial^\mu \alpha$)

Solution: introduce an auxiliary field ξ

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \underline{- \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2} \underline{- A^\mu \partial_\mu \xi}$$

E.o.M. for ξ : $\underline{\partial_\mu A^\mu = 0}$

Lagrange multiplier

E.o.M. for A^μ:

$$\left[-p^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p^\mu p^\nu\right] A_\mu = J^\nu$$

$\det[\quad] \neq 0 \qquad [\quad]^{\mu\nu} p_\nu = \left(-\frac{1}{\xi}\right) p^\mu$

E.v.

Can ~~be~~ inverted

Result reads:

$$\Pi^{\mu\nu} = - \frac{g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2}$$

Check: $\left[-p^2 g^{\mu\alpha} + \left(1-\frac{1}{\xi}\right)p^\mu p^\alpha\right] \cdot \frac{-g_\mu{}^\beta + (1-\xi)\frac{p_\mu p^\beta}{p^2}}{p^2}$

$$= g^{\mu\alpha}\left(g_\mu{}^\beta - (1-\xi)\frac{p_\mu p^\beta}{p^2}\right)$$

$$+\left(1-\frac{1}{\xi}\right)\frac{\cancel{-p^\alpha p^\beta} + (\cancel{1}-\xi)\,p^\alpha p^\beta}{p^2}$$

$$= g^{\alpha\beta} + \frac{p^\alpha p^\beta}{p^2}\cancel{\left(\underbrace{-(1-\xi) - \xi\left(1-\frac{1}{\xi}\right)}_{=0}\right)}$$

Feynman propagator for the photon

$$\boxed{i\Pi^{\mu\nu}(p) = \frac{-i}{p^2+i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]}$$

Covariant gauge (ξ-gauge) ξ

Useful gauges (specify ξ)

1). Feynman – 't Hooft gauge $\xi = 1$

$$i\Pi^{\mu\nu}(p) = \frac{-i\, g^{\mu\nu}}{p^2 + i\varepsilon}$$

2). Lorentz gauge $\xi = 0$

$$i\Pi^{\mu\nu}(p) = \frac{-i}{p^2 + i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$$

3). Unitary gauge $\xi = \infty$

Quite useful for weak bosons $Z, W^\pm$

For massive spin-1

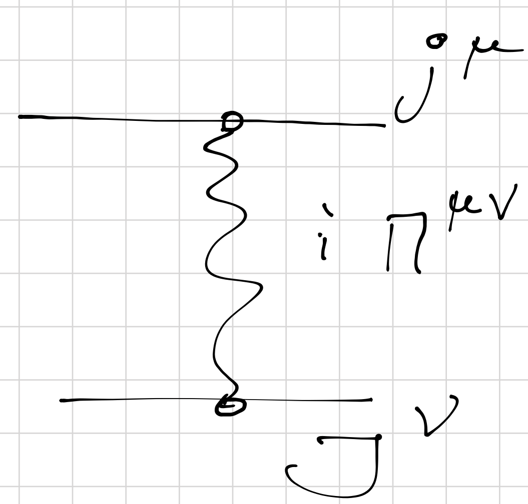
$$\varepsilon_L^\mu = \left(\frac{p}{m}, 0, 0, \frac{E}{m}\right) \xrightarrow[E \to \infty]{} \frac{E}{m}(1, 0, 0, 1)$$

$$\left|\varepsilon_L^\mu M_\mu\right|^2 \sim \left(\frac{E}{m}\right)^2$$

ξ – dependenc $\longrightarrow \sim \xi \dfrac{p^\mu p^\nu}{p^2}$

Gauge invariance : nothing should depend on ξ

Physical amplitudes :



$$A \sim j_\mu J_\nu \cdot i\Pi^{\mu\nu}$$

$$\begin{cases} p_\mu \cdot j^\mu = 0 \\ p_\nu \cdot J^\nu = 0 \end{cases}$$

Is gauge invarianc physical ?

It is not !

A global symmetry is physical

Complex scalar, $\underline{\varphi \to e^{-i\alpha}\varphi}$

↳ Noether current (conserve # of part. − antipart.)

Gauge invariance $\varphi \to e^{-i\alpha(x)}\varphi$

→ redundancy of imposing the locality upon Lagrangian of ∞-range interaction

A toy model

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 (A_\mu + \partial_\mu \pi)^2$$

Gauge-inv. $\left| \begin{array}{l} A^\mu \to A^\mu + \partial^\mu \alpha \\ \pi \to \pi - \alpha \end{array} \right.$

→ For a particular choice of α we can set $\pi = 0$ everywhere

3 d.o.f. − 1 (gaug inv.)

→ 2 indep. polarizations

+ 1 d.o.f. ⟷ π

Now : can integrate π from our theor

Substitute π w. its E.o.M.

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \pi)} = m^2 \partial_\mu \left(A^\mu + \partial^\mu \pi\right) = 0$$

$$\partial_\mu^2 \pi = -\partial_\mu A^\mu \qquad \pi = -\frac{\partial_\mu A^\mu}{\Box}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}\left(1 + \frac{m^2}{\Box}\right) F^{\mu\nu}$$

What's wrong with it?

As you go to large distances

$$\frac{m^2}{\Box} \sim r^2 m^2$$

Typically you expect interactions to vanish as $r \to \infty$

Assigning physics content to G.I. leads to non-locality

A good read: M. Schwartz

QFT and Standard Mod

Chapter 8.6