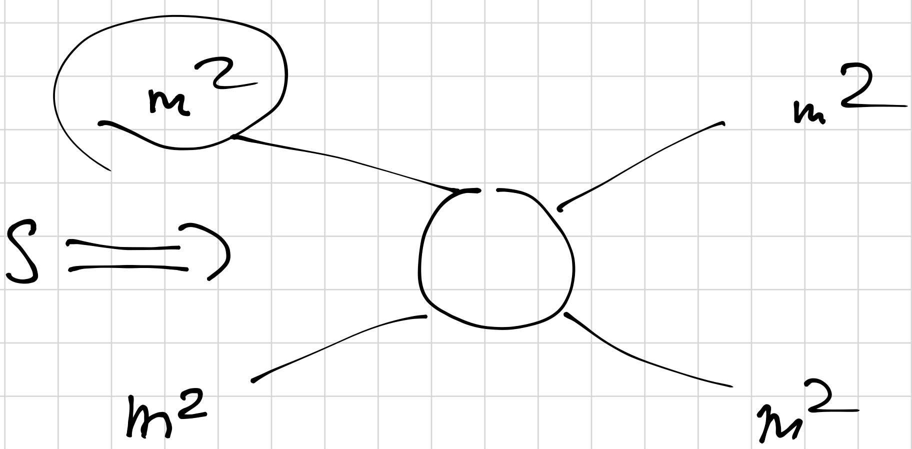
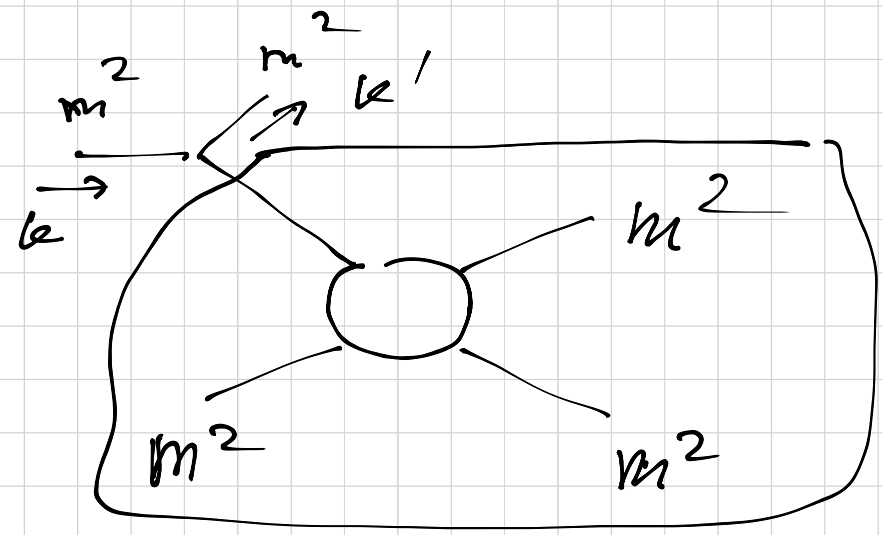
Lecture 2 9/12/2020

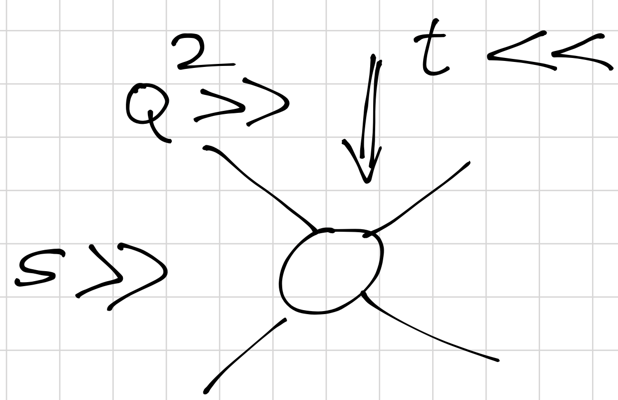


$s \gg m^2$



$(k-k')^2 = -Q^2 < 0$

limit: $\frac{Q^2}{s} = \text{const}$



$$\ln^n\left(\frac{Q^2}{s}\right) \longleftrightarrow \ln^n\left(\frac{s}{m^2}\right)$$

$\frac{Q^2}{s}$ may be small

Quantization of spin-1 field

The task: find irreducible unitary representation of Poincare group for spin-1 field

Spin J $\longrightarrow$ 2J+1 elements

We did $J=0 \longrightarrow$ 1 field

$J=1 \longrightarrow$ 3 components

One could think of a 3-vector

Lorentz invariance $\longrightarrow$ 4-vectors

Conflict: unitarity, norm $\leftrightarrow$ Lorentz inv.

$|\psi\rangle \longrightarrow P|\psi\rangle$

$P(\Lambda, a)$

$\Lambda^{\mu\nu} x_\nu = x^\mu$

$\langle\psi|\psi\rangle = 1$

$V_\mu V^\mu = V^{0\,2} - \vec{V}^2$

$P^\dagger P = \mathbb{1} \longrightarrow \langle\psi|P^\dagger P|\psi\rangle = 1$

Start writing down the Lagrangian

Massive spin-1 field

Spin-0 $\quad \mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) + \frac{1}{2}m^2\varphi^2$

A guess for spin-1:

$$\varphi(x) \longrightarrow A^\mu(x)$$

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2} m^2 A_\mu A^\mu$$

E.o.M.
$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = 0$$

$$\hookrightarrow (\Box + m^2) A^\nu = 0, \qquad \Box = \partial_\mu \partial^\mu$$

$$\mathcal{E} = T^{00} = \frac{\partial \mathcal{L}}{\partial(\dot{A}_\nu)} \dot{A}_\nu - g^{00} \mathcal{L}$$

$$\mathcal{L} = -\frac{1}{2}\left[(\partial^0 A_\nu)(\partial^0 A^\nu) - (\vec{\partial} A_\nu)(\vec{\partial} A^\nu)\right] + \frac{m^2}{2}(A^{0\,2} - \vec{A}^2)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{A}^\nu} = -\partial^0 A^\nu$$

$$\mathcal{E} = \underbrace{-\frac{1}{2}\left[(\partial^0 A^0)^2 + (\vec{\nabla} A^0)^2 + m^2 A^{0\,2}\right]}$$
$$+ \frac{1}{2}\left[(\partial^0 \vec{A})^2 + (\nabla_i A_j)^2 + m^2 \vec{A}^2\right]$$

$\mathcal{E} \rightarrow$ not positive definite
this theory does not have a well-defined spectrum

Consider two other terms $\sim \partial^2$

$$A_\mu \partial^\mu \partial_\nu A^\nu \qquad A_\nu \partial^\mu \partial_\mu A^\nu$$

$$\mathcal{L} = \frac{1}{2} A_\mu \partial^\mu \partial^\nu A_\nu \cdot a + \frac{1}{2} A_\nu \partial^\mu \partial_\mu A^\nu \cdot b$$
$$+ \frac{1}{2} m^2 A_\mu^2$$

E.o.M. $0 = a(\partial_\mu \partial^\mu) A^\nu + b\,\partial_\nu(\partial^\mu A_\mu) + m^2 A^\nu$

Take a derivative ∂_ν

$$\Rightarrow \left[(a+b)\Box + m^2\right](\partial_\nu A^\nu) = 0$$

$a = -b$

$m \neq 0$ $\boxed{\partial_\nu A^\nu = 0}$ $\rightarrow$ removes 1 d.o.f.

we are down to 3 instead of 4

D.o.f. we removed $\longrightarrow$ spin-0 component

Set $a = -b = 1$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu^2$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\downarrow$

E.o.M. $\begin{cases} (\Box + m^2) A^\mu = 0 \\ \partial_\mu A^\mu = 0 \end{cases}$

Proca Lagr.

Energy density

$$\varepsilon = T^{00} : \quad T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \partial^\nu A_\alpha - g^{\mu\nu} \mathcal{L}$$

$$= -F^{\mu\alpha} \partial^\nu A_\alpha + g^{\mu\nu}\left(\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} m^2 A_\mu^2\right)$$

E. and M. fields:

$$\vec{E} = \partial^0 \vec{A} - \vec{\partial} A^0 \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

$$\mathcal{E} = T^{00} = -\left(\partial^0 A^\alpha - \partial^\alpha A^0\right)\partial^0 A_\alpha$$

$$-\frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) - \frac{1}{2} m^2 A_\mu^2$$

$$= \frac{1}{2}\left(\vec{B}^2 + \vec{E}^2\right) + \vec{\partial} A^0\left(\partial^0 \vec{A} - \vec{\partial} A^0\right) - \frac{1}{2} m^2 \left(A^{02} - \vec{A}^2\right)$$

$$= \frac{1}{2}\left(\vec{B}^2 + \vec{E}^2\right) + \frac{1}{2} m^2 \left(A^{02} + \vec{A}^2\right)$$

$$+ \underbrace{A^0 \partial^0 \left(\partial_\mu A^\mu\right)}_{\to 0} - \underbrace{A^0\left(\Box + m^2\right) A^0}_{\to 0} + \underbrace{\partial_i \left(A_0 F^{0i}\right)}_{\downarrow}$$

by E.o.M.

total spatial der. contributes to $\mathcal{E}$

But $E = \int d^3\vec{x}\, \mathcal{E}$

Last term does not contribute to the total energy

$$\mathcal{E} = \frac{1}{2}\left(\vec{B}^2 + \vec{E}^2\right) + \frac{1}{2} m^2 \left(A^{02} + \vec{A}^2\right)$$

This Lagrangean can correspond to a physical theory.

How do we quantize it?

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[\hat{a}_p e^{-ipx} + \hat{a}_p^{+} e^{ipx}\right]$$

$\updownarrow$

$$\left(\Box + m^2\right)\phi = 0 \qquad\qquad \omega_p = \sqrt{\vec{p}^2 + m^2}$$

$$\partial_\mu A^\mu = 0 \qquad \left(\Box + m^2\right) A^\nu = 0$$

$$A^\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_k \left[\hat{a}_p^k \varepsilon_k^\mu(p) e^{-ipx} + \hat{a}_p^{k+} \varepsilon_k^{\mu *}(p) e^{ipx} \right]$$

For $\partial_\mu A^\mu$ to hold : $\boxed{\varepsilon_\mu p^\mu = 0}$

$$p^\mu = (\omega_p, 0, 0, p)$$

$\hookrightarrow$

$$\left.\begin{aligned} \varepsilon_1^\mu(p) &= (0,1,0,0) \\ \varepsilon_2^\mu(p) &= (0,0,1,0) \end{aligned}\right] \quad \text{transverse} \quad \vec{\varepsilon}_i \vec{p} = 0$$

$$\varepsilon_3^\mu(p) = \frac{1}{m}(p, 0, 0, \omega_p) \quad \text{longitudinal}$$

$$\varepsilon_i^\mu \cdot \varepsilon_i^{\mu *} = -1$$

This defines an irreducible represent.
(basis)

Basis depends on $p \to$ infinite-dim. representation

This Lagrangian exists in nature
$\longrightarrow$ describes the heavy W, Z bosons
$M_W \sim 80$ GeV , $M_Z \sim 90$ GeV

$$\varepsilon_L^\mu \underset{\omega \to \infty}{\sim} \frac{\omega}{m}(1, 0, 0, 1)$$

Cross sections $\sim \frac{\omega^2}{m^2} \to \infty$

$$g \frac{E}{M_Z} \sim 1$$

For typical $g \sim 0.1$
$E \sim \frac{M_Z}{g} \sim 1$ TeV

Perturbative unitarity violation
In the Standard Model → cured by Higgs

What happens for $m=0$?

Naively $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cancel{\frac{1}{2}m^2 A_\mu^2}$ $\to 0$

$$\begin{cases} (\Box + m^2)A^\nu = 0 \\ m^2(\partial_\mu A^\mu) = 0 \end{cases} \longrightarrow \begin{array}{l} \Box A^\nu = 0 \\ m=0 \quad \partial_\mu A^\mu \text{ removed} \end{array}$$

Are we back to 4 indep. components?

Gauge symmetry:

$\mathcal{L}$ inv. under $A_\mu \to A_\mu + \partial_\mu \alpha$

E.o.M. $\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$

Gauge fixing: Coulomb gauge $\boxed{\vec{\nabla}\tilde{A} = 0}$

$\vec{\nabla}\vec{A} + \vec{\nabla}^2\alpha = 0$

$$A^0 : (\cancel{\partial^{0\,2}} - \vec{\partial}^2)A^0 - \partial^0(\cancel{\partial^0 A^0} - \underbrace{\vec{\partial}\vec{A}}_{=0}) = 0$$

$$\boxed{\vec{\partial}^2 A^0 = 0}$$

$$\vec{A} : (\partial^{0\,2} - \vec{\partial}^2)A^i - \partial^i(\partial^0 A^0 - \underbrace{\cancel{\vec{\partial}\vec{A}}}_{0}) = 0$$

Since under gauge transf. $A_\mu \to A_\mu + \partial_\mu\alpha$

Coulomb gauge is preserved for any α

such that $\vec{\partial}\vec{A} \to \vec{\partial}\vec{A} + \vec{\partial}^2\alpha$, $\vec{\partial}^2\alpha = 0$

$A^0 \to A^0 + \partial^0 \alpha$ and $\vec{\partial}^2 A^0$

$\hookrightarrow$ we can remove A^0

$\underline{A^0 = 0}$

$$\begin{cases} \Box \vec{A} = 0 \\ \vec{\nabla} \vec{A} = 0 \end{cases}$$

There will be only 2 components $\varepsilon^\mu_{1,2}$

$p^\mu = (p, 0, 0, p)$ $\qquad \varepsilon_{1,2} = \begin{matrix} (0,1,0,0) \\ (0,0,1,0) \end{matrix}$

Circular pol. $\frac{1}{\sqrt{2}}(\varepsilon_1 \pm i \varepsilon_2)$

$$\underline{\varepsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)}$$

What happens in Lorentz gauge $\partial_\mu A^\mu = 0$?

$\varepsilon_L = \frac{1}{\underline{m}}(p, 0, 0, p) \to \infty$

$\to \varepsilon_L = (1, 0, 0, 1)$ $\qquad \varepsilon^\mu_L p_\mu = 0$

$\varepsilon \sim p_\mu$

$$A^\mu_L(x) \sim p^\mu \int \frac{d^3\vec{p}}{(2\pi)^3} \cdots \left(\underline{e^{-ipx}} \cdots \underline{e^{+ipx}} \right)$$

$$= \underline{\partial^\mu}(\ldots)$$

Just a gauge transf. $\to$ unphysical

We want to combine spin-0 and spin-1

Free $\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2$
$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

Interaction $+eA^\mu\underbrace{\varphi\,\partial_\mu\varphi}_{!}$

Not gauge invariant !

$A^\mu\varphi\,\partial_\mu\varphi \longrightarrow A^\mu\varphi\,\partial_\mu\varphi + \underbrace{(\partial^\mu\alpha)\,\varphi\,\partial_\mu\varphi}_{\neq 0}$

We may try to impose some transf. on φ — but there's nothing to combine

No Lagr. with real scalar field and gauge-inv. massless spin-1

$\rightarrow$ Try for complex scalars !

$$\mathcal{L} = -\partial_\mu\varphi^*\partial^\mu\varphi + m^2\varphi^*\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$+\,eA^\mu\varphi^*\partial_\mu\varphi$$

Observe $\underline{\varphi \rightarrow e^{-i\alpha(x)}\varphi}$ $\rightarrow$ symmetry ?

$\varphi \rightarrow e^{-i\alpha}\varphi$ certanly is !

$$A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha$$

Define the covariant der.

$$D^{\mu} = (\partial_{\mu} + ieA_{\mu})\varphi \rightarrow$$

$$\rightarrow (\partial_{\mu} + ieA_{\mu} + i\partial_{\mu}\alpha)\, e^{-i\alpha(x)}\varphi$$

$$= e^{-i\alpha(x)}(\partial_{\mu} + ieA_{\mu})\varphi$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_{\mu}\varphi)^{*}(D^{\mu}\varphi) - m^2|\varphi|^2$$

Scalar QED Lagrangian

In general $D^{\mu} = \partial^{\mu} - ieQA^{\mu}$

$Q \rightarrow$ charge in units of e, $e^{-} \rightarrow Q = -1$

Gauge symmetry and conserved current

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_{\mu}\varphi^{*}\partial^{\mu}\varphi - m^2|\varphi|^2$$

$$+ ieA^{\mu}(\varphi\partial_{\mu}\varphi^{*} - \varphi^{*}\partial_{\mu}\varphi) + e^2A_{\mu}^2|\varphi|^2$$

E.o.M $\quad (\Box + m^2)\varphi = -2ieA_{\mu}\partial^{\mu}\varphi + e^2A_{\mu}^2\varphi$

$$(\Box + m^2)\varphi^{*} = + \text{---}''\text{---} + \text{---}''\text{---}$$

Conserved Noether current

1. Global symmetry $\alpha(x) = \text{const.}$

$$\varphi \rightarrow e^{-i\alpha}\varphi = (1 - i\alpha + \dots)\varphi$$

$$J^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\varphi)} \cdot \frac{\delta\varphi}{\delta\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\varphi^{*})}\,\frac{\delta\varphi^{*}}{\delta\alpha}$$

$$\frac{\delta \varphi}{\delta \alpha} = -i\varphi \qquad \frac{\delta \varphi^*}{\delta \alpha} = +i\varphi^*$$

$$J^\mu = \underbrace{\overbrace{-i\left(\varphi\, \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi\right)}^{J^\mu_{free}}}_{\text{Noether current of the free theory}} - 2eA^\mu |\varphi|^2$$

$$\mathcal{L} = \mathcal{L}_{free} - eA^\mu J_\mu^{free} + e^2 A_\mu^2 |\varphi|^2$$