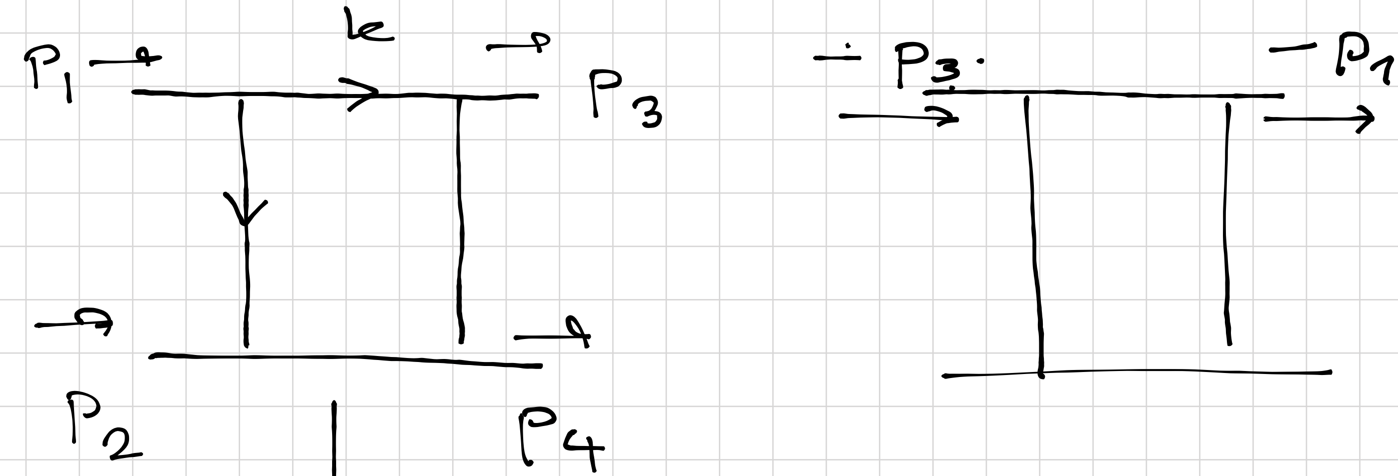
Lecture 11 7/12/20



$$k^\mu = (\alpha, \beta, k_\perp) = \alpha \cdot n^\mu + \beta \bar{n}^\mu + (0, \vec{k}_\perp, 0)$$

$$p_1^\mu \cong (1, \gamma, 0_\perp) \qquad n, \bar{n} = \frac{\sqrt{s}}{2}(1, 0, 0, \pm 1)$$

$$p_2 \cong (\gamma, 1, 0_\perp) \quad \left(\gamma = \frac{m^2}{s}\right) \qquad n^2 = \bar{n}^2 = 0$$

$$2n\bar{n} = s \gg k_\perp^2, m^2$$

Method of regions → determine the positions of poles on complex α-plane (keeping track of signs of $i\varepsilon$)

↳ $\underbrace{\int d^2 k_\perp}_{\hookrightarrow \sim \frac{\pi}{m^2}}$ and $\underbrace{\int d\beta}_{\hookrightarrow \text{ HE asymptotics}}$ splits

At 1-loop level

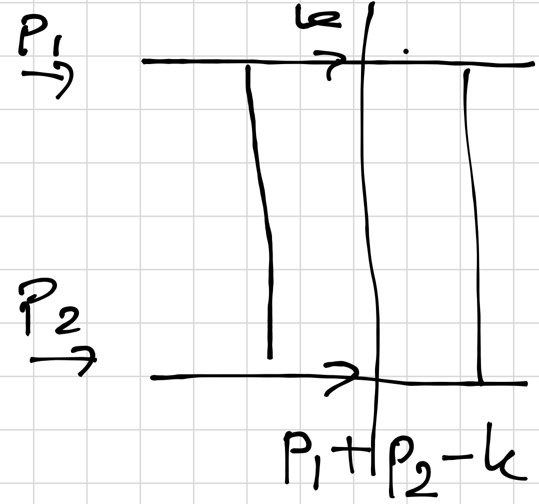
$$A_s^{(4)} \cong -\frac{g^2}{s} \cdot \underbrace{\frac{g^2}{16\pi^2 m^2} \ln\left(-\frac{s}{m^2} - i\varepsilon\right)}_{\ln\left(\frac{s}{m^2}\right) - i\pi \; , \; s > m^2}$$

$$A_s^{(2)} \cong -\frac{g^2}{s}$$

Was obtained also from a DR:

1). Calculate the ln part

→ follows the Cutkosky's rules



$$\rightarrow \frac{1}{k^2-m^2+i\varepsilon}\,\frac{1}{(p_1+p_2-k)^2-m^2+i\varepsilon}$$

$$\rightarrow (-2\pi i)\,\delta(k^2-m^2)\,\Theta(k^0)$$

$$(-2\pi i)\,\delta\big((p_1+p_2-k)^2-m^2\big)\,\Theta(p_1^0+p_2^0-k^0)$$

$$\rightarrow \quad iA_s^{(4)}(s+i\varepsilon) - i\,A_s^{(4)}(s-i\varepsilon)$$

$$= -2\,\mathrm{Im}\,A_s^4(s)$$

$$\hookrightarrow \mathrm{Im}\,A_s^{(4)}(s) = \frac{g^2}{s}\,\frac{g^2}{16\pi m^2}$$

$$\hookrightarrow \mathrm{DR} \quad \mathrm{Re}\,A_s^{(4)}(s) = \frac{1}{\pi}\,\mathcal{P}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\,\mathrm{Im}\,A_s^{(4)}(s')$$

$$\mathcal{P}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\cdot\frac{1}{s'} = -\frac{1}{s}\ln\left(\frac{s}{m^2}\right)$$

$$A_s^{(4)}(s) = \frac{1}{\pi}\int\frac{ds'}{s'-s-i\varepsilon}\,\mathrm{Im}\,A_s^{(4)}(s')$$

$$\left[A_s^{(n)} = -\frac{g^2}{s}\cdot\frac{1}{n!}\left[\frac{g^2}{16\pi^2 m^2}\ln\left(-\frac{s}{m^2}-i\varepsilon\right)\right]^n\right.$$

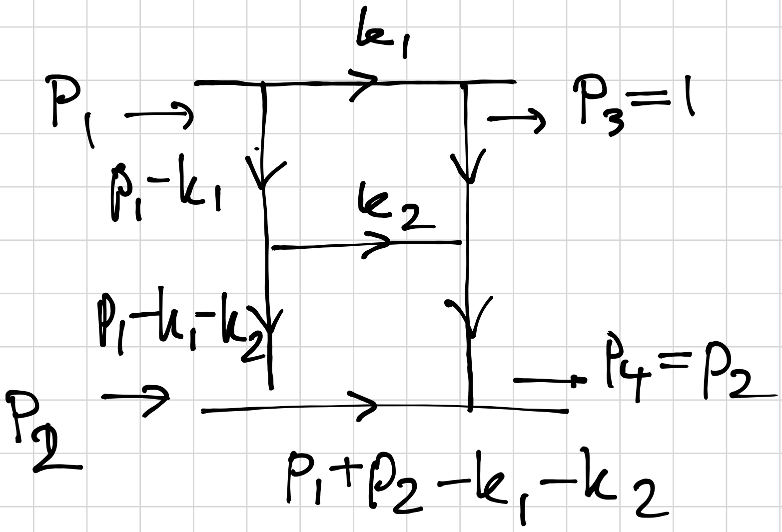
$$\sum_{n=0}^{\infty} A_s^{(n)} = -\frac{g^2}{s}\exp\left[\frac{g^2}{16\pi^2 m^2}\ln\left(-\frac{s}{m^2}-i\varepsilon\right)\right]$$

$$= -\frac{g^2}{s}\cdot\left(-\frac{s}{m^2}-i\varepsilon\right)^{\beta}$$

$$\underline{\underline{\beta}} = \frac{g^2}{16\pi^2 m^2}$$

$\hookrightarrow$ Generalize our result to higher order

Let's obtain $A_s^{(6)} \sim g^6$



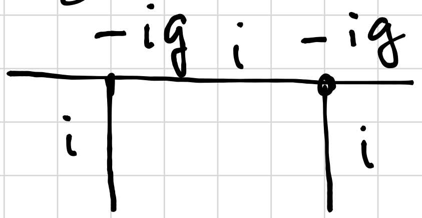
$(3m, \vec{0}) \to$ threshold

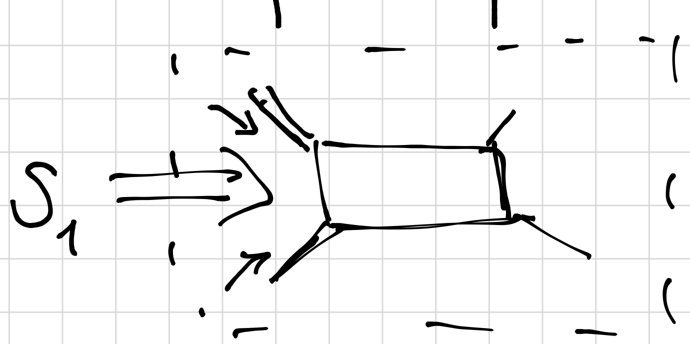
$$= \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{k_1^2 - m^2 + i\varepsilon} \frac{1}{k_2^2 - m^2 + i\varepsilon} \underline{\underline{\frac{1}{(p_1 + p_2 - k_1 - k_2)^2 - m^2 + i\varepsilon}}}$$

$$\frac{1}{\left[(p_1 - k_1)^2 - m^2 + i\varepsilon\right]^2}$$

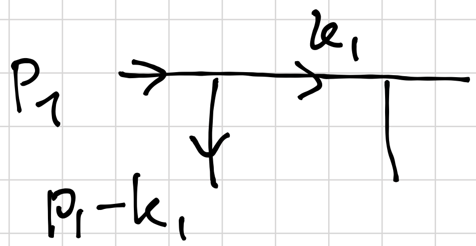
$$\underline{\underline{\frac{1}{\left[(p_1 - k_1 - k_2)^2 - m^2 + i\varepsilon\right]^2}}}$$

Here we only look asymptotics $s \to \infty$
We will use the cutting rules again
and we use the previously obtained
$A_s^{(4)}$ as input

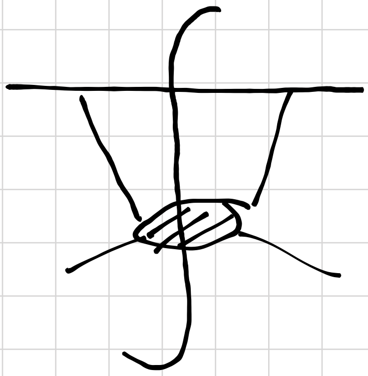




$\longrightarrow A^{(4)}$ at a different energy !



$$s_1 = (p_1+p_2-k_1)^2$$



$$-2\,\mathrm{Im}\,A_s^{(6)}(s) = (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4}\, i^3$$

$$\times(-2\pi i)\,\underline{\delta(k_1^2-m^2)}\,\theta(k_1^0)$$

$$\left[i A_s^{(4)}(s_1+i\varepsilon) - i A_s^{(4)}(s_1-i\varepsilon)\right] \longrightarrow -2\,\mathrm{Im}\,\underline{A_s^{(4)}(s_1)}$$

$$\left[\frac{1}{(p_1-k_1)^2-m^2+i\varepsilon}\right]^2$$

$$\Downarrow$$

$$\mathrm{Im}\,A_s^{(6)}(s) = \frac{g^6}{(2\pi)^3\,16\pi m^2}\int \frac{dk_1^0\, k_1^{\cancel{2}}\, dk_1\, d\Omega}{\underline{s_1}\; 2\cancel{k_1}}\; \frac{\delta\left(k_1-\sqrt{k_1^{0\,2}-m^2}\right)}{\left[(p_1-k_1)^2-m^2+i\varepsilon\right]^2}$$

c.m. $(p_1+p_2)^\mu = (\sqrt{s},\vec{0})$ $\qquad \theta(k_1^0)\,\theta(s_1-4m^2)$

$$s_1 = (p_1+p_2-k_1)^2 = s + \underbrace{k_1^2}_{=\,m^2} - 2\sqrt{s}\,k_1^0$$

$$\underline{k_1^0 = \frac{s-s_1+m^2}{2\sqrt{s}}} \xrightarrow{\delta} k_1 = \sqrt{\left(\frac{s-s_1+m^2}{2\sqrt{s}}\right)^2 - m^2}$$

$$= \frac{\sqrt{\left[s-(\sqrt{s_1}+m)^2\right]\left[s-(\sqrt{s_1}-m)^2\right]}}{2\sqrt{s}}$$

$$\hookrightarrow \quad \underline{\sqrt{s} > \sqrt{s_1}+m \geq 3m} \qquad \Bigg|\quad dk_1^0 = \frac{-ds_1}{2\sqrt{s}}$$

$$\underline{\theta\left(\sqrt{s}-\sqrt{s_1}-m\right)}$$

$$\Rightarrow \frac{g^6}{(2\pi)^3\,16\pi m^2}\,\frac{1}{2}\int^{(\sqrt{s}-m)^2} \frac{ds_1}{s_1}\int d\Omega\,\frac{k_1}{2\sqrt{s}\left[(p_1-k_1)^2-m^2+i\varepsilon\right]^2}$$

$$= \qquad = \quad 4m^2$$

$$1.\ (p_1 - k_1)^2 - m^2 = m^2 - 2p_1 k_1 \approx -\frac{s-s_1}{2}(1-\cos\theta)$$

$s \gg m^2$
$s_1 \gg m^2$

$$\frac{k_1}{2\sqrt{s}} \simeq \frac{s-s_1}{4s}$$

$$p_1 = (1,0,0,1)\frac{\sqrt{s}}{2}$$
$$k_1 = \frac{s-s_1}{2\sqrt{s}}(1, \hat{k}(\theta,\varphi))$$

$$\Downarrow$$

$$\frac{g^6}{(2\pi)^3\, 16\pi m^2}\,\frac{1}{2}\,\frac{1}{4s}\int_{4m^2}^{(\sqrt{s}-m)^2} \frac{ds_1}{s_1}\cancel{(s-s_1)} \underbrace{\int \frac{d\varphi\, d\cos\theta}{\left[\frac{s-s_1}{2}(1-\cos\theta)+m^2\right]^2}}_{2\pi\, \frac{2}{\cancel{s-s_1}}\left(\frac{1}{m^2} - \cancel{\frac{1}{s-s_1+m^2}}\right)}$$

$$= \frac{g^6}{16\pi^2\, 16\pi m^2}\,\frac{1}{m^2}\,\frac{1}{s}\int_{4m^2}^{(\sqrt{s}-m)^2} \frac{ds_1}{s_1}$$

$$= \frac{g^2}{s}\cdot\frac{g^2}{16\pi m^2}\cdot\frac{g^2}{16\pi^2 m^2}\cdot \ln\frac{s}{m^2} + \text{corr.}$$

$$\mathrm{Re}\, A_s^{(6)} = \frac{1}{\pi}\, P\int_{9m^2}^{\infty} \frac{ds'}{s'-s}\, \mathrm{Im}\, A_s^{(6)}(s')$$

$$= g^2\left(\frac{g}{4\pi m}\right)^4 \underbrace{P\int \frac{ds'}{s'(s'-s)}\ln\left(\frac{s'}{m^2}\right)}_{\frac{1}{s}\, P\int_{9m^2}^{\infty}\left(\frac{ds'}{s'-s} - \frac{ds'}{s'}\right)\ln\frac{s'}{m^2}}$$

Sub. $x = \frac{gm^2}{s'}$ $\longrightarrow$ $P\int \ldots \longrightarrow -P\int_0^1 \frac{dx}{x-y} \ln\frac{1}{x}$

$y = \frac{gm^2}{s}$

$\downarrow$

$$\mathrm{Re}\, A_s^{(6)} = -\frac{g^2}{s}\left(\frac{g}{4\pi m}\right)^4 \underline{\mathrm{Li}_2\left(\frac{s}{m^2}+i\varepsilon\right)} + \ldots$$

Dilogarithm function

$$\mathrm{Li}_2(y) = -\int_0^1 \frac{dt}{t} \ln(1-yt)$$

Properties of Li_2:

1. $\mathrm{Li}_2(0) = 0$
2. Real for $-\infty < y < 1$; branch cut for $y > 1$
3. $\mathrm{Li}_2(1) = \frac{\pi^2}{6}$

Identities:

$$\begin{cases} \mathrm{Li}_2\left(\frac{1}{x}\right) + \mathrm{Li}_2(x) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-x) \\ \mathrm{Li}_2(1-x) + \mathrm{Li}_2(x) = \frac{\pi^2}{6} - \ln x \ln(1-x) \end{cases}$$

$$\mathrm{Li}_2\Big(\underbrace{\frac{s}{m^2}+i\varepsilon}_{x}\Big) = \mathrm{Li}_2\left(\frac{m^2}{s}\right) - \frac{\pi^2}{6} - \frac{1}{2!}\ln^2\left(-\frac{s}{m^2}-i\varepsilon\right)$$

$$\sim \frac{m^2}{s} \to 0$$

$\Downarrow$

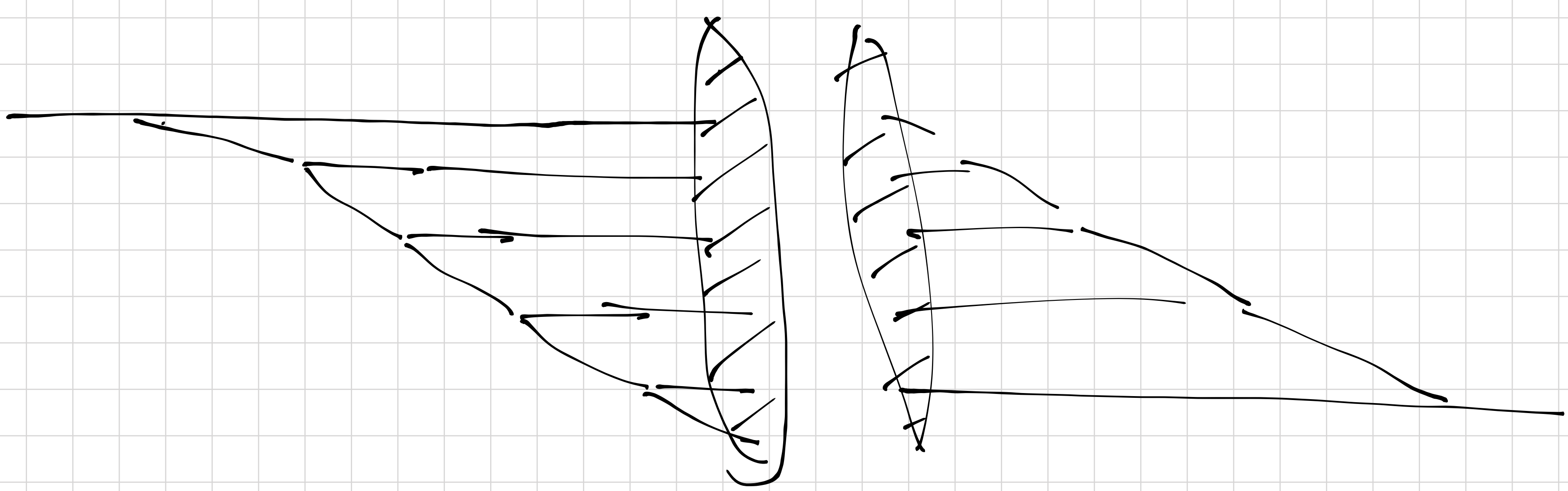
$$\boxed{A_s^{(6)}(s) \simeq -\frac{g^2}{s}\frac{1}{2!}\left(\frac{g}{4\pi m}\right)^4 \ln^2\left(-\frac{s}{m^2}-i\varepsilon\right)}$$

$$A_s^{(2n)} \cong -\frac{g^2}{s}\cdot\frac{1}{2(n-1)!}\left(\frac{g}{4\pi m}\right)^{2n} \ln^{2(n-1)} \ln\left(-\frac{s}{\mu^2}-i\varepsilon\right)$$

$$\operatorname{Im} A^{(n+2)} = g^2\int \frac{d^4k_1}{(2\pi)^4}\frac{\delta(k_1^2-\mu^2)\,\theta(\dots)}{\left[(p_1-k_1)^2-\mu^2+i\varepsilon\right]^2}$$

$$\cdot \operatorname{Im} A^{(n)}(s_1)$$

At high energies the interaction between the cloud (shower) of virtual particles



Unitarity of the S-matrix

$S \approx e^{-iHt}$ $\qquad S_{fi} = \langle f|U(t_2\to+\infty, t_1\to-\infty)|i\rangle$

$S^\dagger S = \mathbb{1}$

$S_{fi} \to (S-\mathbb{1})_{fi}$

$S-\mathbb{1} = iT$

$S^\dagger S = 1 = (\mathbb{1}-iT^\dagger)(\mathbb{1}+iT) = \mathbb{1}$

$\longrightarrow i(T^\dagger - T) = T^\dagger T$

$$\langle f | i(T^\dagger - T) | i \rangle = i(2\pi)^4 \delta^4(p_f - p_i) \times \left[A^*_{f\to i} - A_{i\to f} \right]$$

$$T_{fi} = (2\pi)^4 \delta^4(p_f - p_i)\, A_{fi}$$

$$\langle f | T^\dagger | i \rangle = \langle i | T | f \rangle^*$$

$$\langle f | T^\dagger T | i \rangle = \sum_X \int \prod_{j\in X} \frac{d^3\vec{p}_j}{(2\pi)^3 2E_j} \langle f | T^\dagger | X \rangle \langle X | T | i \rangle$$

$$\mathbb{1} = \sum_X \int d\Pi_X \, | X \rangle \langle X |$$

$$\Downarrow$$

$$\boxed{A_{i\to f} - A^*_{f\to i} = i \sum_X \int d\Pi_X (2\pi)^4 \delta^4(p_i - p_X)\, A_{i\to X} A^*_{f\to X}}$$

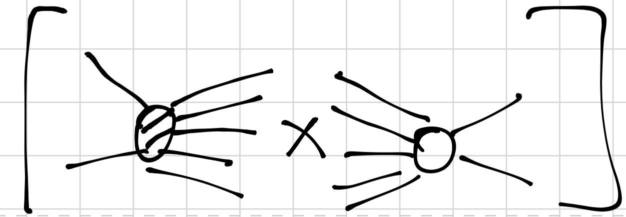
Important case: $i = f$ forward scattering $2 \to 2$

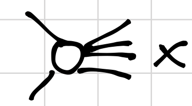
$$A_{i\to i} - A^*_{i\to i} = 2i A.$$

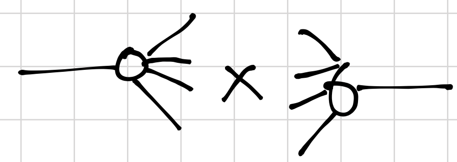
r.h.s. $\rightarrow$ $i \underbrace{\sum_X \int d\Pi_X (2\pi)^4 \delta^4(p_i - p_X) | A_{i\to X} |^2}_{\displaystyle \equiv 4 E_{cm} |\vec{p}|\, \sigma_{tot}}$

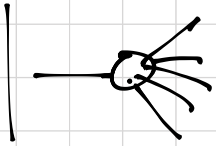
Optical theorem

$$\boxed{\mathrm{Im}\, A_{i\to i} = 2 E_{cm} |\vec{p}|\, \sigma_{tot}}$$

$|i\rangle \rightarrow$ 1-particle state $\quad \mathrm{Im}$ 

$$\mathrm{Im}\, A_{i\to i} = M_i \cdot \Gamma_{tot} \sim \left| \right.$$  $$\left. \right|^2$$

$$\mathrm{Im}\left[\right.$$  $$\left.\right]$$

$$\sim \left|\right.$$  $$\left.\right|^2 \sim \Gamma_{tot}$$

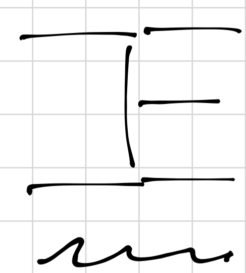
Resonance $M_R \to M_R - \frac{i\Gamma_R}{2}$

$$\mathrm{Im}\left(p^2 - M_R^2\right) \to \mathrm{Im}\left(p^2 - \left(M - \frac{i\Gamma}{2}\right)^2\right) = M\Gamma$$

$$\mathrm{Im}\, A \sim 2E_{cm}\, p\, \sigma_{tot}$$

$$A = \sum_{n=0}^{\infty} c_n \frac{1}{n!} g^n \qquad c_n = \text{complex fn.}$$

$$\mathrm{Im}\, A^{(n)} = \frac{1}{n!} g^n c_n = 2E_{cm}\, p\, \sigma^{(n)}$$

$$A^{(6)} \to g^6 \left|\right.$$  $$\left.\right|^2$$

Loop amplitudes can be reconstructed from tree-level amplitudes

$$A^{(6)} \sim \int d\Pi_X\, A^{(2)} \cdot A^{(4)}$$

Connects lower order to higher orders
$\to$ we used it for ladder resummation