Lecture 10

$$A^{(4)} \cong -\frac{g^2}{s}\left(\frac{g}{4\pi m}\right)^2 \ln\left(-\frac{s}{m^2} - i\varepsilon\right)$$

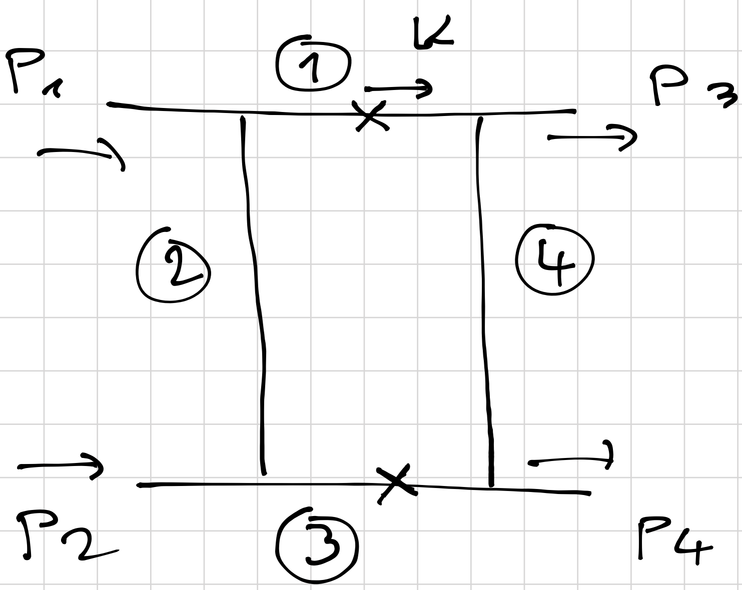
$$\ln\left|\frac{s}{m^2}\right| - i\pi$$

$$\mathrm{Im}\, A^{(4)} = -\frac{g^2}{s}\frac{g^2}{16\pi m^2}$$

$$\hookrightarrow \int_0^1 \frac{d\beta}{\underbrace{\beta s - m^2 - i\varepsilon}} \sim -\frac{1}{s}\ln\left(\frac{-s}{m^2} - i\varepsilon\right)$$

Im part $\longrightarrow$ pick the pole

$$\frac{1}{x - x_0 - i\varepsilon} = \mathcal{P}\frac{1}{x - x_0} + i\pi\,\delta(x - x_0)$$



$$\alpha = \alpha_3$$

$$k^\mu = (\alpha, \beta, \vec{k}_\perp) \equiv \alpha \cdot n^\mu + \beta \bar{n}^\mu + \vec{k}_\perp$$

Cutcosky's rules:

For each propagator

$$\frac{1}{p^2 - m^2 + i\varepsilon} \longrightarrow \underline{(-2\pi i)\,\delta(p^2 - m^2)\,\Theta(p^0)}$$

$$A^{(4)}(s + i\varepsilon) - A^{(4)}(s - i\varepsilon) = -2i\,\mathrm{Im}\, A^{(4)}(s)$$

$$-2i\,\mathrm{Im}\,A^{(4)}(s) = +ig^4\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(2)\,(4)}$$

$$\cdot\left(-2\pi i\,\underline{\delta(k^2-m^2)}\,\Theta(k^0)\right)$$

$$\cdot\left(-2\pi i\,\delta((p_1+p_2-k)^2-m^2)\,\Theta(p_1^0+p_2^0-k^0)\right)$$

$$\mathrm{Im}\,A^{(4)}(s) = \frac{g^4}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(2)\,(4)}\,(2\pi)^2\times\{\dots\}$$

C.M. $\left|\begin{array}{l} p_1^\mu+p_2^\mu = (\sqrt{s},\vec{0}) \\ \delta((p_1+p_2-k)^2-m^2) = \delta(s-2\sqrt{s}\,k^0) \\ \delta(k^2-m^2) = \delta(k^{0\,2}-\vec{k}^2-m^2) \end{array}\right.$

$$\{\dots\} = \frac{1}{2\sqrt{s}}\,\delta\!\left(k^0-\frac{\sqrt{s}}{2}\right)\Theta(\sqrt{s}-k^0)$$

$$\cdot\ \frac{1}{2|\vec{k}|}\,\underbrace{\delta\!\left(|\vec{k}|-\sqrt{\frac{s}{4}-m^2}\right)}_{\Theta(\sqrt{s}-2m)}\,\Theta(k^0)$$

$$= \frac{g^4}{2}\,\frac{1}{(2\pi)^2}\int dk^0\,\vec{k}^2\,d|\vec{k}|\,d\Omega\ \frac{1}{(2)\,(4)}$$

$$\frac{1}{4|\vec{k}|\sqrt{s}}\,\delta\!\left(k^0-\frac{\sqrt{s}}{2}\right)\delta\!\left(|\vec{k}|-\sqrt{\frac{s}{4}-m^2}\right)$$

$$\Theta(k^0)\,\Theta(\sqrt{s}-k^0)\,\Theta(\sqrt{s}-2m)$$

$$= \frac{g^4}{32\pi^2} \frac{|\vec{k}|}{\sqrt{s}} \underline{\Theta(\sqrt{s}-2m)}$$

$$\int \frac{d\Omega}{\left[(p_1-k)^2-m^2\right]\left[(p_3-k)^2-m^2\right]}$$

Forward limit: $\quad p_1 = p_3$

$$p_1^\mu = \left(\frac{\sqrt{s}}{2}, 0, 0, \sqrt{\frac{s}{4}-m^2}\right)$$

$$k^\mu = \left(\frac{\sqrt{s}}{2}, \sqrt{\frac{s}{4}-m^2}\ \hat{k}(\theta, \varphi=0)\right)$$

$$(p_1-k)^2 = -2|\vec{k}|^2(1-\cos\theta)$$

$$d\Omega = \int_{-1}^{1} d\cos\theta \underbrace{\int_0^{2\pi} d\varphi}_{\to 2\pi}$$

$$\Rightarrow \frac{g^4}{16\pi} \frac{|\vec{k}|}{\sqrt{s}} \int_{-1}^{1} \frac{d\cos\theta}{\left[2\vec{k}^2(1-\cos\theta)+m^2\right]^2}$$

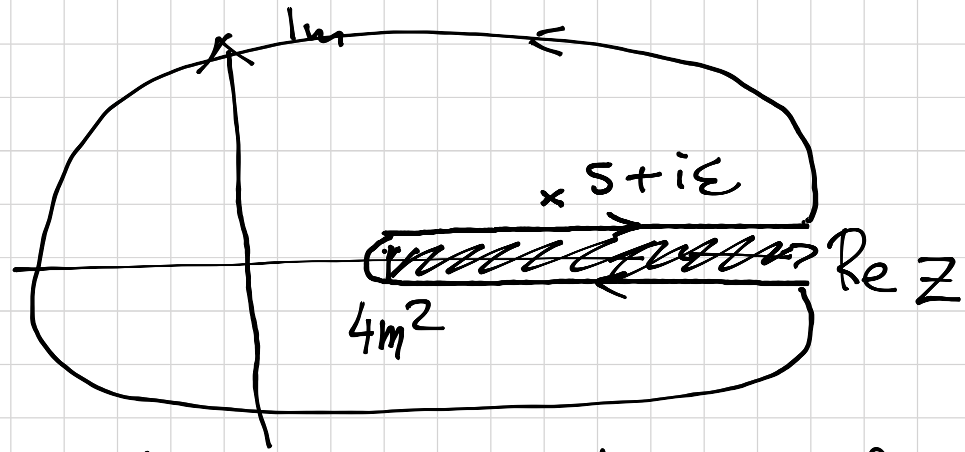
$$= \frac{g^4}{16\pi} \frac{1}{2|\vec{k}|\sqrt{s}} \left[\frac{1}{m^2} - \frac{1}{4k^2+m^2}\right]$$

$$= \frac{g^4}{16\pi m^2} \frac{1}{2|\vec{k}|\sqrt{s}} \frac{4\vec{k}^2}{4\vec{k}^2+m^2} \longrightarrow \boxed{\frac{g^2}{s} \frac{g^2}{16\pi m^2}}$$

$$s \gg m^2$$

The same result for $\operatorname{Im} A^{(4)}$ as before

$A^{(4)}$



Re part $\longrightarrow$ can be reconstructed from Cauchy's theorem

$$\oint_C dz\, A^{(4)}(z) = 0$$

$$\oint \frac{dz}{z-s-i\varepsilon}\, A^{(4)}(z) = 2\pi i\, A^{(4)}(s+i\varepsilon)$$

$$= \int_{4m^2}^{\infty} \frac{dz}{z-s-i\varepsilon} \Big[\underbrace{A^{(4)}(z+i\eta) - A^{(4)}(z-i\eta)}_{= \, 2i\, \mathrm{Im}\, A^{(4)}(z)} \Big]$$

$$A^{(4)}(s+i\varepsilon) = \frac{1}{\pi} \int \underbrace{\frac{dz}{z-s-i\varepsilon}}_{\downarrow \; P\frac{1}{z-s} + i\pi\delta(z-s)} \mathrm{Im}\, A^{(4)}(z)$$

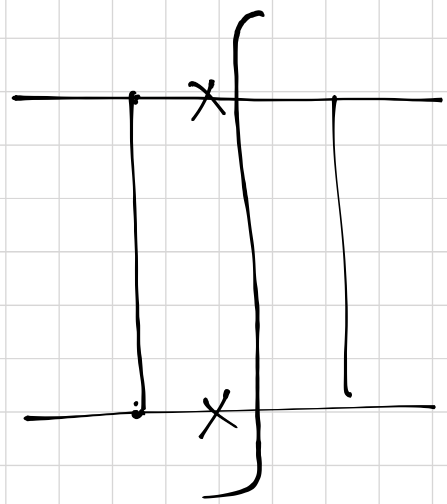
$$A^{(4)}(s+i\varepsilon) = \left[\mathrm{Re}\, A^{(4)} + i\, \mathrm{Im}\, A^{(4)}\right](s)$$

$$\mathrm{Re}\, A^{(4)}(s) = \frac{1}{\pi} P \int \frac{dz}{z-s} \boxed{\mathrm{Im}\, A^{(4)}(z)}$$

$$\mathrm{Re}\, A^{(4)}(s) = g^2 \left(\frac{g}{4\pi m}\right)^2 P \int_{4m^2}^{\infty} \frac{dz}{z(z-s)}$$

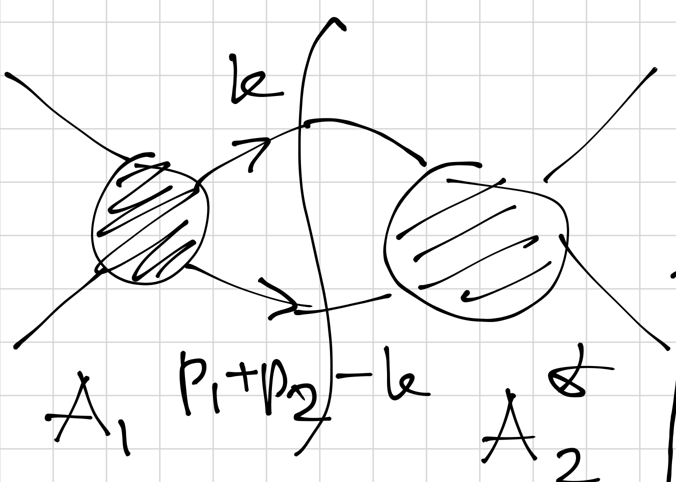
$$P\int \frac{dz}{z(z-s)} = \frac{1}{s}\left[P\int_{4m^2}^{\infty}\frac{dz}{z-s} - \int_{4m^2}^{\infty}\frac{dz}{z}\right]$$

$$= -\frac{1}{s}\ln\left(\frac{s}{m^2}\right) + \underbrace{\dots}_{O\left(\frac{m^2}{s}\right)}$$



1). From 4d $\int$ by method of regions

2). Using cutting (Cutkosky's) rules



$$\mathrm{Re}\,A = \frac{1}{\pi}P\int \frac{ds'}{s'-s}\,\mathrm{Im}\,A$$

$$\mathrm{Im}\,A \sim \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,A_1\cdot A_2^* \cdot (2\pi)^2\,\delta(k^2-m^2)\,\delta\left((p_1+p_2-k)^2-\right)$$

dispersion relation

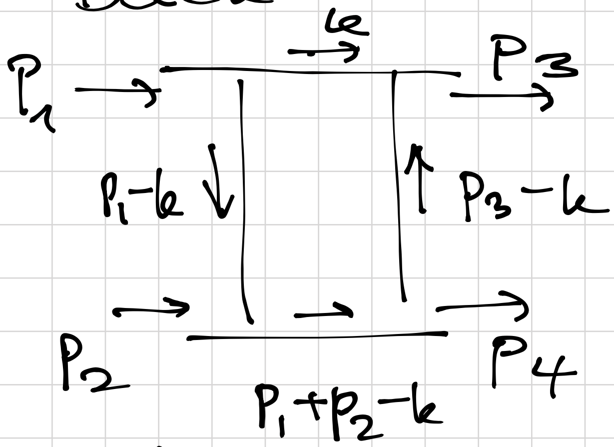
In ϕ^3 the DR converges

$$\left(\mathrm{Im}\,A \sim \frac{1}{s}\right)$$

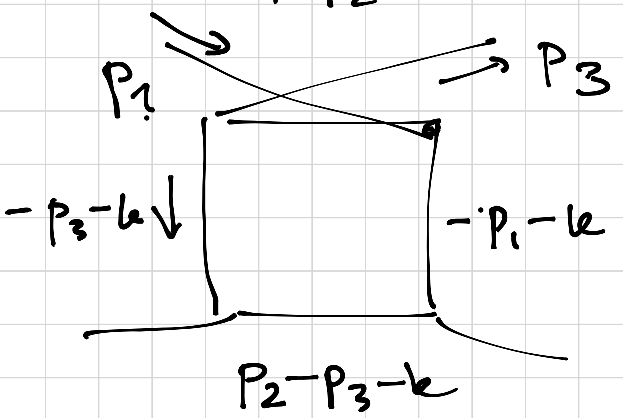
DR may have to be subtracted

$$\mathrm{Re}\,A(s) - \mathrm{Re}\,A(s_0) = \frac{s-s_0}{\pi}\int \frac{ds'}{(s'-s)(s'-s_0)}\,\mathrm{Im}\,A(s')$$

Back to boxes and ladders



$$\cong -\frac{g^2}{s}\left(\frac{g}{4m\pi}\right)^2 \ln\left(-\frac{s}{m^2}-i\varepsilon\right)$$

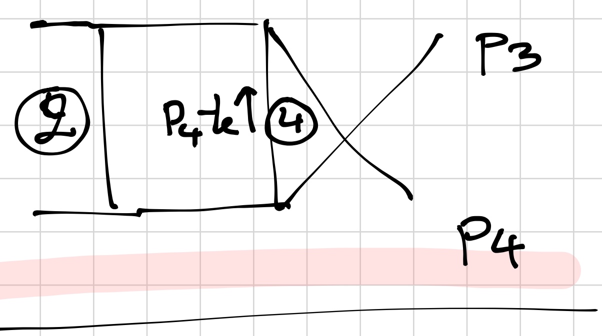


$$(p_2-p_3)^2 = u$$

$$p_1 \leftrightarrow -p_3$$

$$k_\perp^2 = (-k_\perp)^2 \qquad \underline{-u \sim s}$$

$$\cong -\frac{g^2}{u}\left(\frac{g}{4\pi m}\right)^2 \ln\left(-\frac{u}{m^2}-\cancel{i\varepsilon}\right)$$

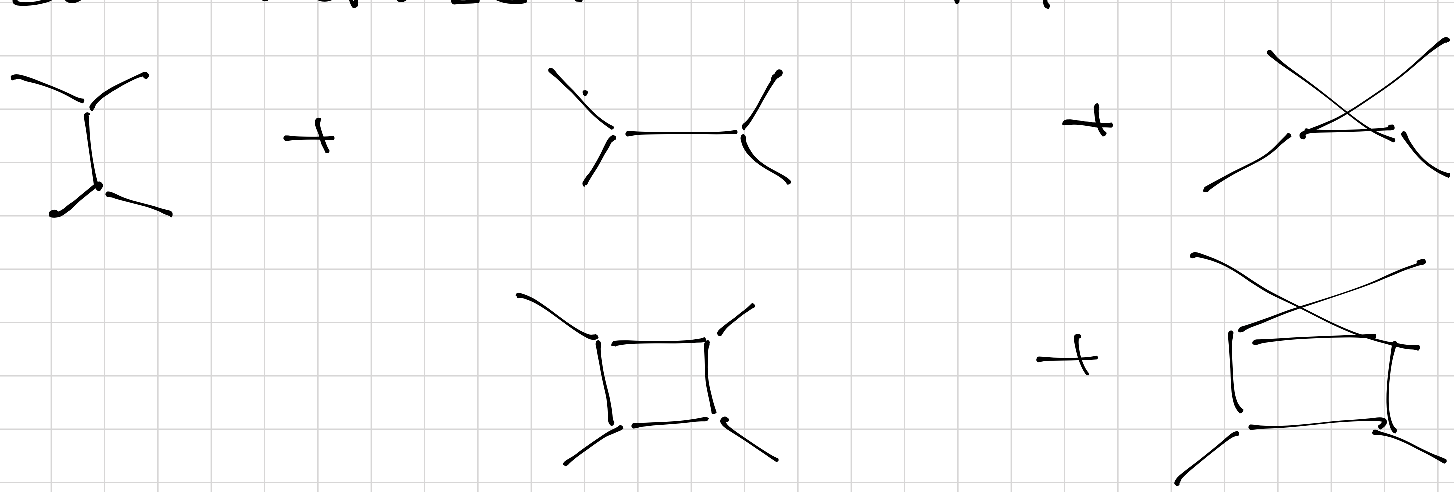


$$t \leftrightarrow u$$

$$\int \frac{d^2k_\perp}{(2)(4)} \sim \int \frac{d^2k_\perp}{k_\perp^2+m^2}\,\frac{1}{(-u+\dots)} \hookrightarrow \frac{1}{s}$$

Will not matter

To summarize: $2^{nd} + 4^{th}$

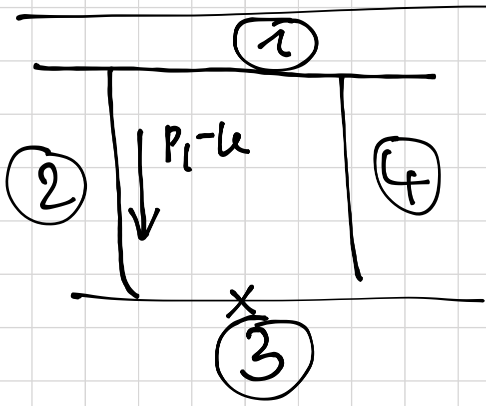


$$\cong -\frac{g^2}{t-m^2} - \frac{g^2}{s}\left(1+\left(\frac{g}{4\pi m}\right)^2 \ln\left(-\frac{s}{m^2}-i\varepsilon\right)\right)$$

$$-\frac{g^2}{u}\left(1+\left(\frac{g}{4\pi m}\right)^2 \ln\left(-\frac{u}{m^2}-i\varepsilon\right)\right)$$

$$\sum_n \left[\left(\frac{g}{4\pi m}\right)^2 \ln\left(-\frac{s}{m^2} - i\varepsilon\right)\right]^n \frac{1}{n!}$$

$$= \left(-\frac{s}{m^2}\right)^{\underline{\alpha(g, m^2, t)}}$$



$(\sum p_i)^2 - m^2 \sim \alpha, \beta, k_\perp^2$

picked a pole at 3

$\alpha_3 = 1 + \gamma$

2 : $p_1 - k = (-\gamma, \gamma - \beta, -\bar{k}_\perp)$

1 : $k = (1 + \gamma, \beta, \bar{k}_\perp)$

4 : $p_3 - k \simeq (-\gamma, \gamma - \beta, \bar{\Delta}_\perp - k_\perp)$

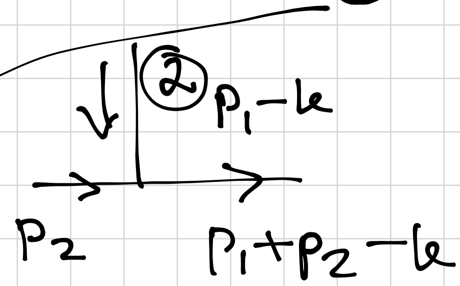
All quantities are small except for β

$\gamma = \frac{m^2}{s} \ll \beta \ll 1$

$\underline{(p_1 - k)^2} = -\gamma(\gamma - \beta)s - k_\perp^2 \sim -m^2$

2 4 $\sim (-\gamma, -\beta, -\bar{k}_\perp)$

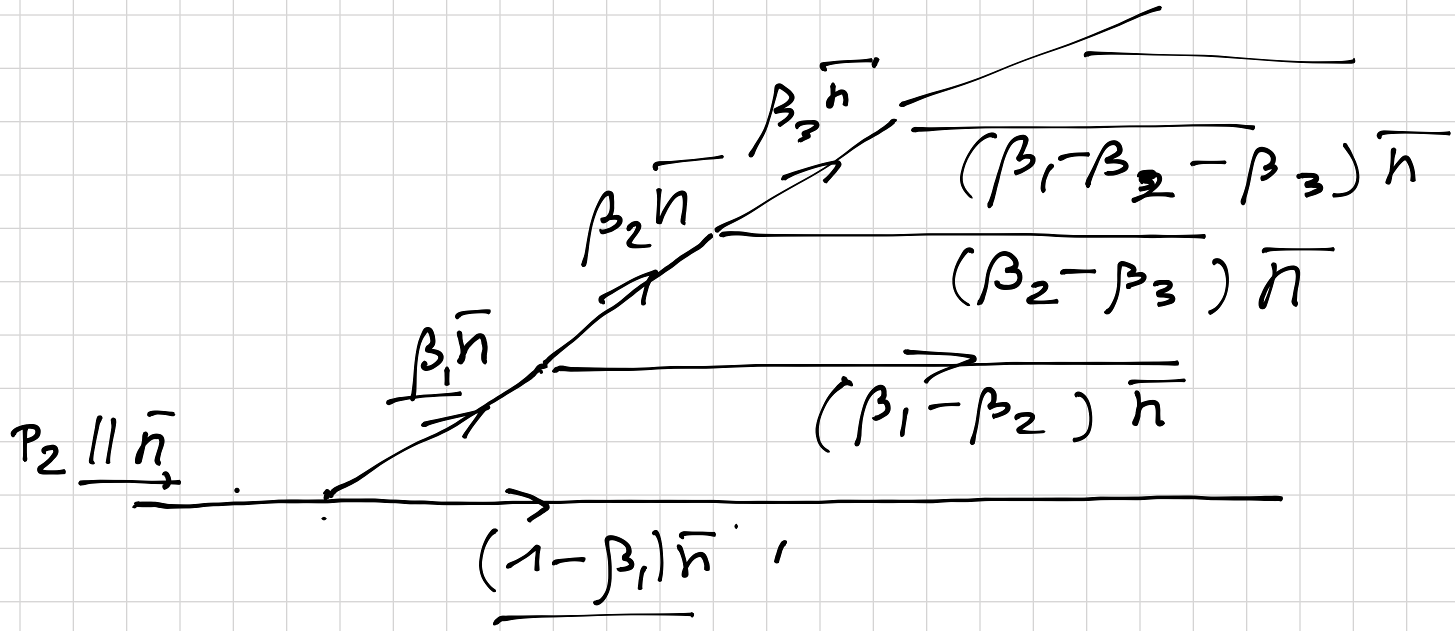
$p_1^\mu \simeq (1, \gamma, 0_\perp)$ $\qquad p_2^\mu = (\gamma, 1, 0_\perp)$



$1 \to \frac{1-\beta}{\beta}$

$\hookrightarrow (1 + \gamma - \alpha_3, 1 + \gamma - \beta$

$E_2 = \frac{\sqrt{s}}{2} \cdot 1$



$$\int_\gamma^1 \frac{d\beta}{\beta} \longrightarrow \int^1 \frac{d\beta_1}{\beta_1} \int \frac{d\beta_2}{\beta_2} \cdots \int_\gamma^{\beta_{n-1}} \frac{d\beta_n}{\beta_n}$$

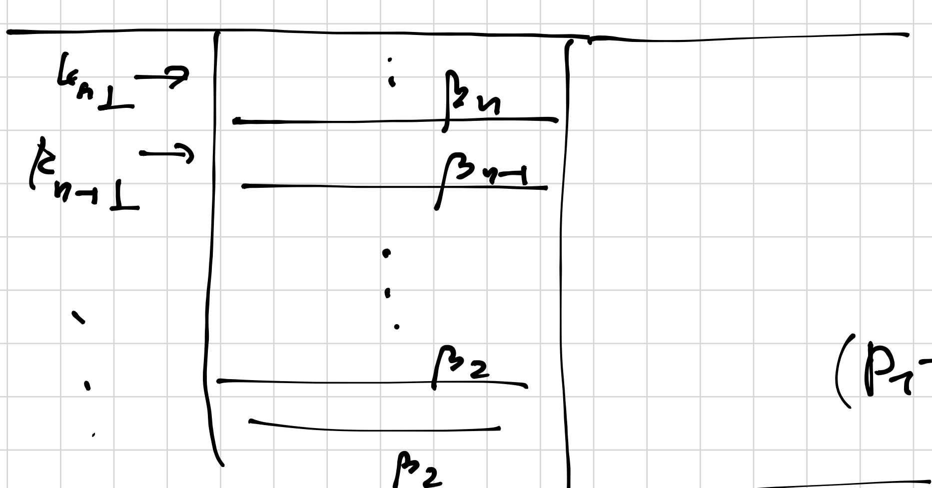
$$\longrightarrow \frac{1}{(n-1)!}\left[\int_\gamma^1 \frac{d\beta}{\beta}\right]^{n-1}$$

Impose strong ordering

$$\frac{m^2}{s} = \gamma \ll \beta_1 \ll \beta_2 \ll \ldots \beta_2 \ll \ldots 1$$

$$E \sim \beta \cdot \frac{\sqrt{s}}{2}$$

$$\frac{m^2}{2\sqrt{s}} \ll E_n \ll E_{n-1} \ll \ldots \ll \frac{\sqrt{s}}{2}$$



$$\int \frac{d\beta_i}{\beta_i}$$

$$(p_1-k_i)^2 \sim -\gamma(\gamma-\beta_i)s - k_{i\perp}^2$$

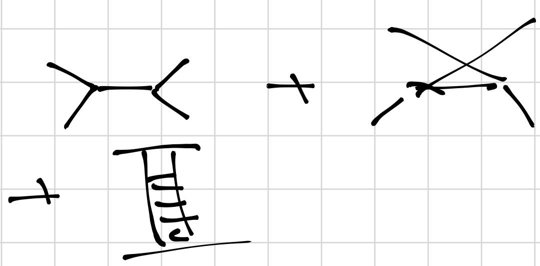
$\ll$

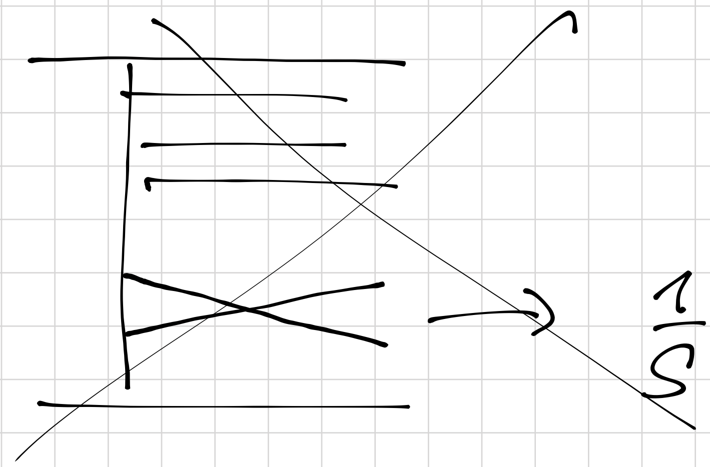
$$\sim -k_{i\perp}^2$$

$$\prod \int \frac{d\beta_i}{\beta_i} \int \frac{d^2 k_{i\perp}}{k_{i\perp}^2 + m^2}$$

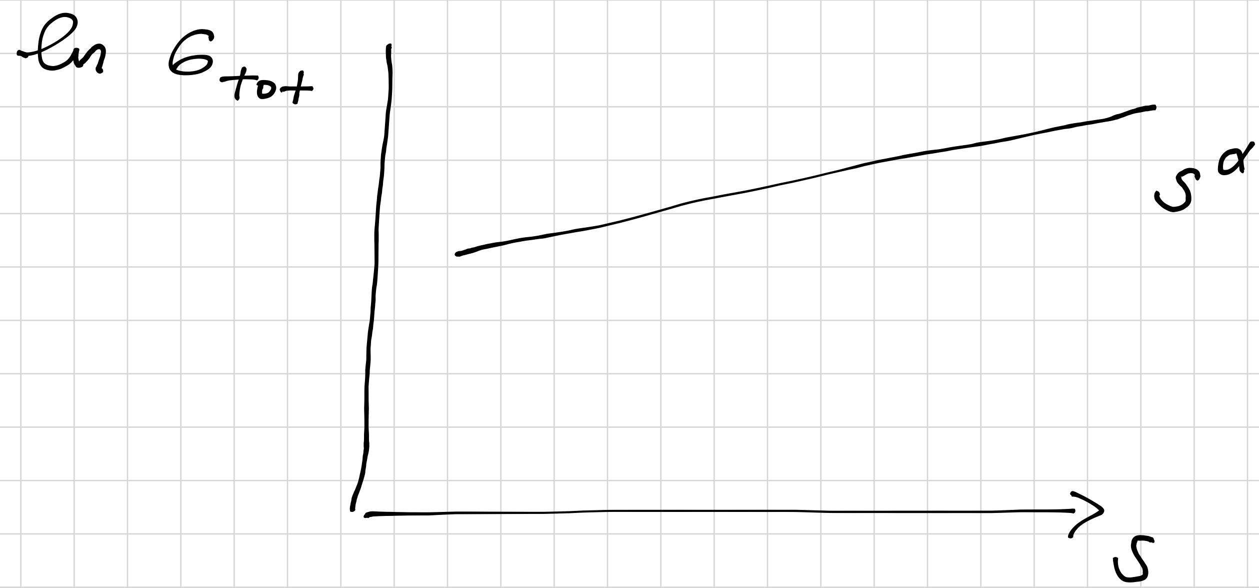
$$A_s^{(n)} = -\frac{g^2}{s}\frac{1}{(n-1)!}\left[\left(\frac{g}{4\pi m}\right)^2 \ln\left(-\frac{s}{m^2} - i\varepsilon\right)\right]^{n-1}$$

$A_u^{(n)}$ —— // —— $(s \to u)$



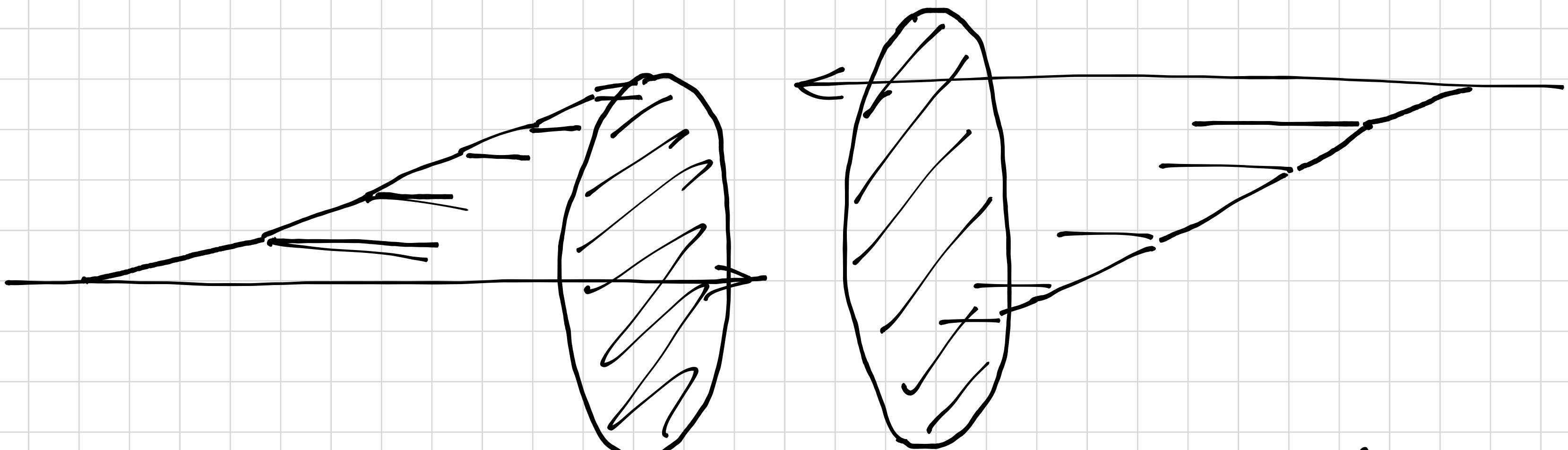


$$A_s + A_u = \frac{g^2}{m^2}\left(-\frac{s}{m^2}\right)^{-1+\alpha(g^2, m^2, t)} + \frac{g^2}{m^2}\left(-\frac{u}{m^2}\right)^{-1+\alpha(g^2, m^2, t)}$$



Physical interpretation

We have 2 ultra-rel. part. traveling w. almost $v \simeq c$ in the opposite dir.



There is no time for them to interact

$$\rightarrow \quad A^{(2)} \sim -\frac{g^2}{s}$$

But can emit a collinear particle

$\hookrightarrow$ this is why in quantum theory the classical or QM geometric interpretation of cross section breaks down!

$\sigma = \pi R^2$ Classical

$\sigma \sim \frac{4\pi R^2}{2\pi R^2}$ QM $4\,(2) \rightarrow$ diffraction