Exercise sheet 5
Theoretical Physics 3 : QM WS2020/2021
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04.12.2020

Exercise 1. Finite square well potential. (20 points)

A particle of mass m is moving in a finite square potential well

$$V(x) = \begin{cases} -\frac{\alpha}{2a} & \text{for } -a \leq x \leq a \\ 0 & \text{for } x > |a| \end{cases}.$$

The energy levels are determined by the condition

$$z \tan z = \sqrt{z_0^2 - z^2}$$

where

$$z = \frac{a}{\hbar}\sqrt{2m\left(E + \frac{\alpha}{2a}\right)}, \qquad z_0 = \frac{a}{\hbar}\sqrt{\frac{m\alpha}{a}}.$$

a) *(10 p.)* Considering the limit $a \to 0$ and assuming that E is finite in this limit, show that you recover the unique bound state of the δ potential well $E = -\frac{m\alpha^2}{2\hbar^2}$.

b) *(10 p.)* What should be the value of $m\alpha a/\hbar^2$ in order for the system to have precisely n bound states?

Exercise 2. (35 + 10 points)

Consider Schrödinger equation with the following potential:

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & 0 < x < a \\ V_1 & x > a. \end{cases}$$

a) *(10 p.)* Consider bound states of the system ($E < 0$). Derive the transcendental equation(s) for the energy quantum number. Notice in the case $V_1 = 0$ one should recover expressions for the finite well problem considered in the lecture.

b) *(10 p.)* Write down the eigenfunctions of the Hamiltonian for the case $0 < E < V_1$. Sketch an eigenfunction for some intermediate value of E.

c) *(5 p.)* Show that in the limit $V_1 \to +\infty$ the eigenfunctions of the Hamiltonian vanish for $x > a$. Is it true that not only the eigenfunctions of the Hamiltonian, but *all* the wave functions must vanish for $x > a$?

d) *(10 p.)* Consider scattering states for the case $E > V_1$. Derive expressions for the reflection and transmission coefficients.

e) *(10 p.)(Bonus)* Consider the limit $V_1 \to +\infty$. Prove that the system admits no bound states if and only if $V_0 \leq \pi^2\hbar^2/(8ma^2)$.

Exercise 3. WKB approximation (30 + 10 points)

Consider the potential $V(x)$ shown on Fig. 1

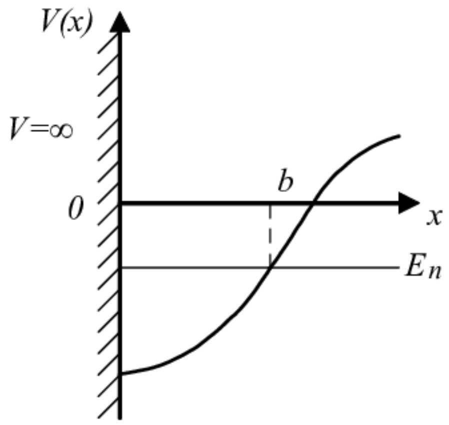

Figure 1

To the right of the right turning point b, the wave function has the semi-classical form:

$$\Psi(x) = \frac{C}{2\sqrt{|p(x)|}} \exp\left(-\frac{1}{\hbar}\int_b^x |p(x)|dx\right), \quad x > b\,. \tag{1}$$

In the region $x < b$ the corresponding wave function is then:

$$\Psi(x) = \frac{C}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar}\int_x^b p(x)dx + \frac{\pi}{4}\right), \quad x < b\,. \tag{2}$$

a) *(5 p.)* Derive the following relation for the energy levels:

$$\frac{1}{\hbar}\int_0^b \sqrt{2m[E_n - V(x)]}dx = \pi\left(n + \frac{3}{4}\right).$$

b) *(5 p.)* From now on (applies to tasks b), c), d) and e)), consider the specific potential:

$$V(x) = \begin{cases} -\frac{\alpha}{x^2} & \text{for } x \geq a \\ \infty & \text{for } x < a \end{cases} \qquad \alpha > 0\,.$$

Under which conditions can the semi-classical approximation be applied? For which values of x?

c) *(10 p.)* Using the result of a) and performing an integration, derive the relation for the energy levels E_n.

d) *(10 p.)* Find the explicit form for the upper energy levels, defined by the condition $|E_n| \ll \alpha/a^2$ expanding the result of c) by the small parameter $|E_n|a^2/\alpha$. Keep the leading terms only. How does the distance between energy levels change with growing n?

e) *(10 p.)(Bonus)*. Obtain the explicit form of the wave functions $\Psi_n(x)$.

Exercise 4. Matrices: eigenvalues and eigenvectors. (15 points)

a) *(2 p.)* Consider three Pauli matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute σ_x^2, σ_y^2, σ_z^2, and the commutators $[\sigma_x, \sigma_y]$, $[\sigma_y, \sigma_z]$, $[\sigma_z, \sigma_x]$.

b) *(3 p.)* Find the eigenvalues and eigenvectors of σ_x, σ_y and σ_z.

c) *(5 p.)* Find the eigenvalues and eigenvectors of the 2D rotation and hyperbolic rotation matrices, correspondingly:

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{bmatrix}.$$

d) *(5 p.)* Find the eigenvalues and eigenvectors of the following 3×3 matrices:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$