Exercise sheet 7
Theoretical Physics 3 : QM WS2020/2021
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Exercise 1. (20 points)

a) *(15 p.)* Show that the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in three dimensions in spherical coordinates takes the form

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

b) *(5 p.)* Show that the radial term can also be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}r.$$

Exercise 2. (45 points)

a) *(15 p.)* Show that $L_\pm Y_l^m = \hbar\sqrt{l(l+1) - m(m\pm 1)}\, Y_l^{m\pm1}$.
Hint: Consider the norm of $L_\pm Y_l^m$.

b) *(15 p.)* Show that for eigenfunctions of $\hat{L}_z$, we have

$$\langle \hat{L}_x\rangle = \langle \hat{L}_y\rangle = 0, \qquad \langle \hat{L}_x^2\rangle = \langle \hat{L}_y^2\rangle \qquad \text{and} \qquad \langle \hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x\rangle = 0.$$

Hint: Consider $\langle \hat{L}_\pm\rangle$ and $\langle \hat{L}_\pm^2\rangle$.

c) *(15 p.)* In the state ψ_{lm} with definite angular momentum l and its z-component m, find the mean values $\langle \hat{L}_x^2\rangle$, $\langle \hat{L}_y^2\rangle$ as well as the mean values $\langle \hat{L}_{\tilde{z}}\rangle$ and $\langle \hat{L}_{\tilde{z}}^2\rangle$ of the angular momentum projection along the $\tilde{z}$-axis making an angle α with the z-axis.
Hint: Use $\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}^2$.

Exercise 3. (35 points)

a) *(20 p.)* Prove that for a particle in a potential $V(\vec{r})$ the rate of change of the expectation value of the orbital angular momentum $\vec{L}$ is equal to the expectation value of the torque (rotational analog to Ehrenfest's theorem)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{L}\rangle = \langle\vec{N}\rangle \qquad \text{where} \qquad \vec{N} = \vec{r}\times(-\vec{\nabla}V).$$

Show that $\langle\vec{L}\rangle$ is constant for any spherically symmetric potential. (This is one form of the quantum statement of conservation of angular momentum.)

b) *(15 p.)* Show that the mean values of the vectors $\vec{L}$, $\vec{r}$, $\vec{p}$ for the particle state with wave function $\psi = \exp(i\vec{p}_0\cdot\vec{r}/\hbar)\phi(\vec{r})$ are connected by the classical relation $\vec{L} = \vec{r}\times\vec{p}$. Here, p_0 is a real vector and $\phi(\vec{r})$ is a real function.

Exercise 4. (*Bonus* 15 points)

Confirm or invalidate the following assertions:

a) *(10 p.)* If $[\hat{H}, \hat{\vec{L}}] = \vec{0}$, the energy levels do not depend on m (i.e., on the eigenvalues of the projection of one of the components of the angular momentum $\hat{\vec{L}}$).

b) *(5 p.)* If $[\hat{H}, \hat{L}^2] = 0$, the energy levels do not depend on l.