Exercise sheet 6
Theoretical Physics 3 : QM WS2020/2021
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Exercise 1. (35 points)

Recall the one-dimensional quantum harmonic oscillator problem with the spectrum given by

$$E_n = (n + \tfrac{1}{2})\hbar\omega$$

and its stationary states $|n\rangle$:

$$\hat{H}|n\rangle = E_n |n\rangle \,.$$

Recall the ladder operators

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar\omega m}}(m\omega\hat{x} \mp i\hat{p}),$$

which act on $|n\rangle$ in the following way:

$$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle \;, \qquad \hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle \;.$$

We know that $|n\rangle$ is a complete orthonormal basis in our Hilbert space. In such *discrete* basis, a vector is expressed as an infinite discrete column of values, and an operator is expressed in terms of an infinite matrix. Consider matrix representations of $\hat{H}$, $\hat{x}$ and $\hat{p}$ operators in this basis.

a) *(5 p.)* Write down the matrix representation of the Hamiltonian $H_{nm} = \langle n|\hat{H}|m\rangle$.

b) *(10 p.)* Without using explicit expression for $\langle x|n\rangle$, show

$$\langle n|\hat{x}|m\rangle = \sqrt{\frac{\hbar}{2m\omega}}\begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{nm}$$

Hint: express $\hat{x}$ in terms of the ladder operators.

c) *(10 p.)* Without using explicit expression for $\langle x|n\rangle$, show

$$\langle n|\hat{p}|m\rangle = -i\sqrt{\frac{\hbar m\omega}{2}}\begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & \cdots \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & -\sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{nm}$$

Hint: express $\hat{p}$ in terms of the ladder operators.

d) *(10 p.)* Given $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$, derive the spatial representation of the Hamiltonian

$$\langle x| \hat{H} |x'\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2}\right)\delta(x-x').$$

Notice when acting on a state in spatial representation we recover the "usual" operator form of the Hamiltonian:

$$\int \mathrm{d}x' \, \langle x| \hat{H} |x'\rangle \langle x'|\psi\rangle = \int \mathrm{d}x' \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2}\right)\delta(x-x') \langle x'|\psi\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2}\right)\langle x|\psi\rangle,$$

where $\langle x|\psi\rangle \equiv \psi(x)$.

Note that all vectors and operators are entities which are invariant across bases. And representations in chosen bases (domains) are simply different ways to express the same entity.

Exercise 2. Hermitian operators (30 points)

In this exercise, you are going to prove some theorems which are very important for quantum mechanics.

a) *(5 p.)* **Theorem I:** If two operators $\hat{A}$ and $\hat{B}$ commute, and if $|\psi\rangle$ is an eigenvector of $\hat{A}$, then $\hat{B}|\psi\rangle$ is also an eigenvector of $\hat{A}$ with the same eigenvalue.

b) *(5 p.)* **Theorem II:** If two observables A and B commute, and if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two eigenvectors of $\hat{A}$ with different eigenvalues, then the matrix element $\langle\psi_1| \hat{B} |\psi_2\rangle$ vanishes.

c) *(15 p.)* **Theorem III:** If two observables A and B commute, one can construct an orthonormal basis made of eigenvectors common to both $\hat{A}$ and $\hat{B}$. Consider only the case where the spectra of A and B are discrete.
Hint: Since A is an observable, there is at least one orthonormal basis made of eigenvectors of $\hat{A}$:

$$\hat{A}\left|u_n^i\right\rangle = a_n \left|u_n^i\right\rangle, \qquad n = 1, 2, \dots \qquad i = 1, 2, \dots, g_n,$$

where g_n is the degree of degeneracy of eigenvalue a_n and with $\langle u_n^i|u_m^j\rangle = \delta_{nm}\delta_{ij}$. Discuss the matrix elements $\langle u_n^i|\hat{B}|u_m^j\rangle$.

d) *(5 p.)* Show the reciprocal of the theorem III.

Exercise 3. Operators and Dirac notation (35 points)

a) *(5 p.)* Consider the ladder operators for the quantum harmonic oscillator problem. Show that

$$\hat{a}_+ = (\hat{a}_-)^\dagger.$$

b) *(5 p.)* Show that for any observable $\hat{q}$ with *nondegenerate* spectrum

$$\hat{q} = \sum_q q\,|q\rangle\langle q|,$$

where $\hat{q}|q\rangle = q|q\rangle$, and in the case of continuous spectrum $\sum_q \to \int \mathrm{d}q$.
Hint: since the set of eigenfunctions is complete and orthonormal, one can use

$$\sum_q |q\rangle\langle q| = \hat{1}.$$

c) *(5 p.)* We already know from the lecture that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px}.$$

Show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\,\langle p|\psi\rangle\,.$$

d) *(5 p.)* Show that

$$\langle\psi|\hat{x}|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}p\,\phi^*(p)\left[i\hbar\frac{\partial}{\partial p}\right]\phi(p),$$

where

$$\phi(p) \equiv \langle p|\psi\rangle$$

e) *(5 p.)* Show that

$$\left\langle x\middle|\hat{p}\middle|x'\right\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x').$$

f) *(10 p.)* Recall the infinite square well stationary state wave function

$$\langle x|n\rangle = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right) & 0 < x < a, \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\langle p|n\rangle$.