# Relativistic QFT (Theo 6a): Exercise Sheet 7
# **Total: 200 points - Christmas bonus**

21/12/2020

## 1. Bhabha and Møller scattering in scalar QED (30 points)

Consider Bhabha (elastic electron-positron) scattering process $e^+e^- \to e^+e^-0$ and Møller (elastic electron-electron) scattering process $e^-e^- \to e^-e^-$ at the $2^{nd}$ order ($\sim e^2$) of perturbation series in scalar QED in the c.m. frame.

(a) Draw all possible Feynman diagrams at this order (tree-level). Using Feynman rules for the scalar QED (see lecture 14 on the website for details), obtain the respective expressions for the invariant amplitudes $A^B(s,t)$ and $A^M(s,t)$ for the tree-level Bhabha and Møller scattering, correspondingly. Observe that the amplitude $A^B$ is symmetric under crossing $s \leftrightarrow t$, while $A^M$ under $u \leftrightarrow t$.

(b) Find the differential cross sections $\frac{d\sigma^B}{d\Omega}$ and $\frac{d\sigma^M}{d\Omega}$ for both processes.

(c) Noting for fixed $s$, the $t$-variable only depends on the scattering angle, derive the differential cross sections with respect to $t$, i.e. show that

$$\frac{d\sigma}{dt} \equiv \frac{d\sigma}{d\Omega}\frac{d\Omega}{dt} = -\frac{1}{16\pi}\frac{|A|^2}{s(t+u)} \tag{1}$$

Determine the limiting values of $t$ and try to find the total cross sections $\sigma^B(s)$ and $\sigma^M(s)$. Why are the integrals divergent?

## 2. Compton scattering in scalar QED (120 points)

The Compton ($e^-\gamma \to e^-\gamma$) scattering amplitude at the $2^{nd}$ order of the perturbation series in scalar QED was derived in the lectures with the result

$$A = e^2\epsilon^{\mu}_{\lambda}(\hat{q})\epsilon^{*\nu}_{\lambda'}(\hat{q}')\left[2g_{\mu\nu} - \frac{(2p+q)_\mu(2p'+q')_\nu}{(p+q)^2-m^2} - \frac{(2p-q')_\mu(2p'-q)_\nu}{(p-q')^2-m^2}\right] = A_{\lambda\lambda'}. \tag{2}$$

The photon *helicity* is $\lambda, \lambda' = \pm 1$. For a photon traveling along the direction $\hat{q}' = (\sin\theta, 0, \cos\theta)$, the respective polarization vector with helicity $\lambda'$ is given by $\epsilon^{\nu}_{\lambda'}(\hat{q}') = \frac{1}{\sqrt{2}}(0, \cos\theta, i\lambda', -\sin\theta)$.

(a) Evaluate $A_{\lambda\lambda'}$ in the laboratory frame, $p^\mu = (M, \vec{0})$, $q^\mu = \omega_L(1,0,0,1)$, $q'^\mu = \omega'_L(1, \sin\theta_L, 0, \cos\theta_L)$. The photon energies are related to the invariants via $\omega_L = \frac{p\cdot q}{m} = \frac{s-m^2}{2m}$ and $\omega'_L = \frac{p\cdot q'}{m} = \frac{m^2-u}{2m}$. Obtain the Compton formula

$$\omega'_L = \frac{\omega_L}{1 + \frac{\omega_L}{m}(1-\cos\theta_L)}. \tag{3}$$

Which terms in the square bracket of Eq.(2) contribute $A_{\lambda\lambda'}$ in the lab frame and which don't?

(b) Repeat the calculation in the center-of-momentum frame ($q^\mu = \omega_{cm}(1,0,0,1)$, $q'^\mu = \omega_{cm}(1, \sin\theta_{cm}, 0, \cos\theta_{cm})$, $p^\mu = (\sqrt{s}, \vec{0}) - q^\mu$, with the photon energy given by $\omega_{cm} = \frac{s-m^2}{2\sqrt{s}}$). Which terms contribute to $A_{\lambda\lambda'}$ now?

(c) To arrive at the (differential) cross section one needs to square the amplitude, sum over the final photon polarizations and average over those of the initial photon, i.e. compute

$$\frac{1}{2}\sum_{\lambda,\lambda'=\pm1}|A_{\lambda\lambda'}|^2 \tag{4}$$

Compute this in the lab. and c.m. frames.

(d) Does the result depend on the reference frame if expressed in terms of invariants? Hint: to answer this question, start from frame-specific expressions and arrive at

$$\frac{1}{2}\sum_{\lambda,\lambda'=\pm1}|A_{\lambda\lambda'}|^2 = 2e^4\left\{1+\left[1-\frac{2m^2t}{(s-m^2)(u-m^2)}\right]^2\right\} \tag{5}$$

(e) Starting from the invariant expression

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s\omega_{cm}^2}\frac{1}{2}\sum_{\lambda,\lambda'=\pm1}|A_{\lambda\lambda'}|^2, \tag{6}$$

work your way towards the differential cross sections in the lab frame. Using the Compton formula (3), express the differential

$$dt = \frac{\omega_L'^2}{\pi}d\Omega_L, \quad d\Omega_L = d\cos\theta_L d\phi, \tag{7}$$

obtain the unpolarized differential cross section in the lab frame

$$\frac{d\sigma}{d\Omega_L} = \frac{\alpha^2}{2m^2}\frac{1+\cos^2\theta_L}{\left[1+\frac{\omega_L}{m}(1-\cos\theta_L)\right]^2}, \quad \alpha = \frac{e^2}{4\pi}. \tag{8}$$

Find the low-energy behavior of the differential cross section.

(f) Do the same as in (e) in the c.m. frame, using

$$dt = \frac{\omega_{cm}^2}{\pi}d\Omega_{cm}, \quad d\Omega_{cm} = d\cos\theta_{cm}d\phi, \tag{9}$$

(g) Derive the total cross section as function of $\omega_L$, integrating (8) over $\theta_L$.

# 3. Boundaries of the scattering regions on the Mandelstam plane (50 points)

Obtain the limiting values $t_{min}$ and $t_{max}$ as functions of other available invariants for the $2 \to 2$ processes listed below. Plot them on the $(s-u,t)$ plane for each case:

(a) Real Compton scattering $\gamma(q)+e^-(p)\to\gamma(q')e^-(p')$ with $q^2=q'^2=0,\ p^2=p'^2=m^2$

(b) Virtual Compton scattering with a *spacelike* initial photon (originating from e.g. an electron scattering process) $\gamma^*(q)+e^-(p)\to\gamma(q')e^-(p')$ with $q^2=-Q^2<0,\ q'^2=0,\ p^2=p'^2=m^2$.

(c) Virtual Compton scattering with a *timelike* initial photon (originating from e.g. $e^+e^-$ annihilation) $\gamma^*(q)+e^-(p)\to\gamma(q')e^-(p')$ with $q^2=Q^2>0,\ q'^2=0,\ p^2=p'^2=m^2$.

(d) Doubly-virtual Compton scattering with *one spacelike and one timelike* photon $\gamma^*(q)+e^-(p)\to\gamma^*(q')e^-(p')$ with $q^2=-q'^2=-Q^2<0,\ p^2=p'^2=m^2$.

Hint: for each reaction write down the constraint on the Mandelstam invariants $s+t+u=\sum_i p_i^2$. Remember that $t_{min}(t_{max})$ are Lorentz invariants and do not depend on the frame. Hence, to find $t_{min}\,(t_{max})$ work in your favorite reference frame and put $\theta=0\,(\theta=\pi)$, respectively.