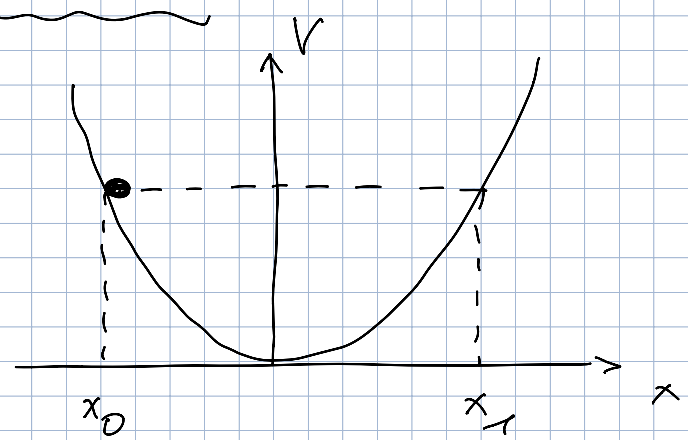
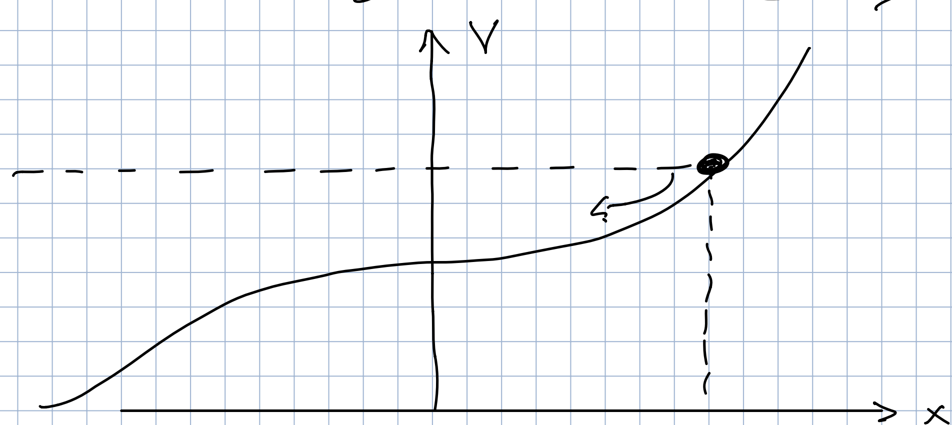
⟹ VORLESUNG 6 QM

2.5 DELTA FUNKTION POTENTIAL

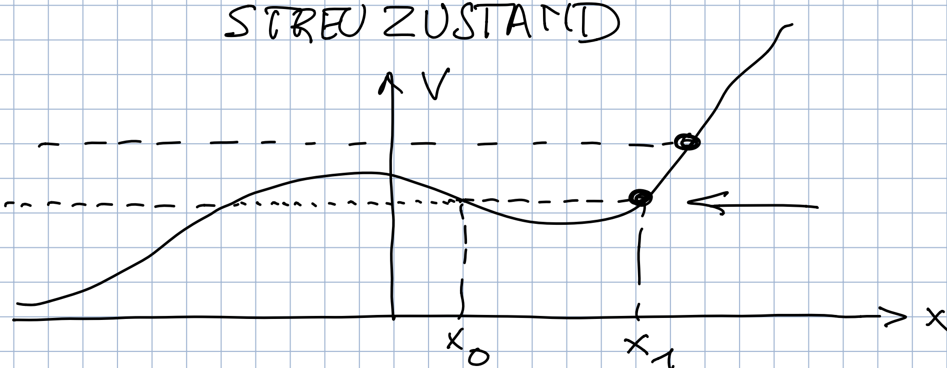
⟹ KLASSISCH



GEBUNDENER ZUSTAND



STREUZUSTAND



$\Rightarrow$ QM

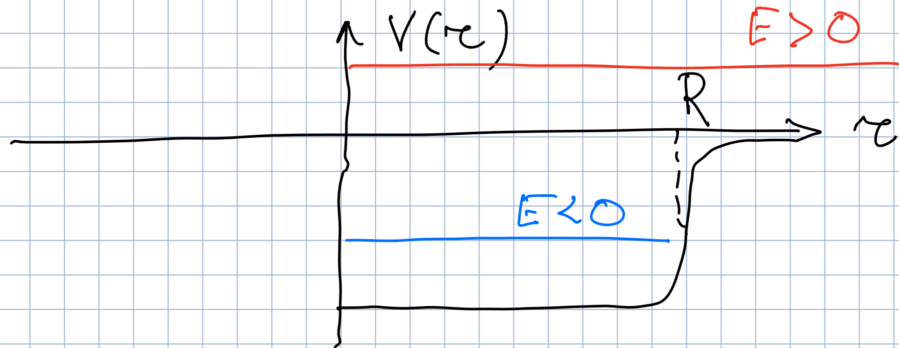
GEBUNDENER ZUSTAND

$$E < V(+\infty) \,\&\, V(-\infty)$$

STREU ZUSTAND

$$E > V(+\infty) \vee V(-\infty)$$

e.g. OFT $V(+\infty) = V(-\infty) = 0$

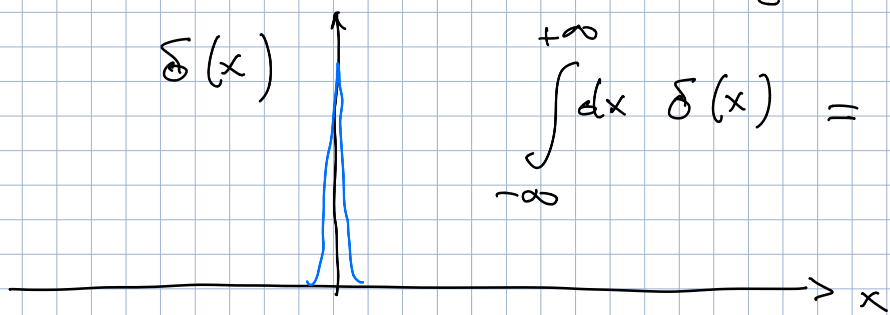


STREUZUSTAND $\Rightarrow$ KONTINUIERLICH k

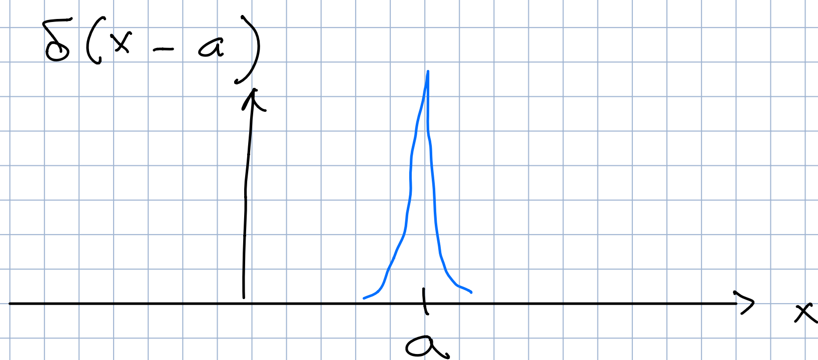
GEBUNDENE ZUSTAND $\Rightarrow$ DISKRETE n

δ - FUNKTION

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$



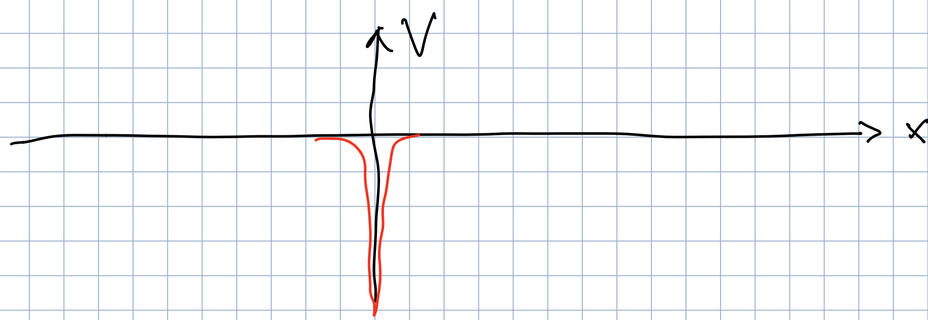
$$\int_{-\infty}^{+\infty} dx \;\delta(x) = 1$$



$$\int_{-\infty}^{+\infty} dx \; f(x)\,\delta(x-a) = f(a)$$

$$= f(a) \underbrace{\int_{-\infty}^{+\infty} dx \;\delta(x-a)}_{1}$$

$$\Rightarrow \quad V(x) = -\alpha\,\delta(x) \qquad \underline{\alpha > 0}$$



$E < 0 \quad \Rightarrow \quad$ GEBUNDENE ZUSTÄNDE

$E > 0 \quad \Rightarrow \quad$ STREU ZUSTÄNDE

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi = E\,\psi \Leftarrow$$

- $\underline{\underline{E < 0}}$ $\qquad E = -\frac{\hbar^2}{2m}K^2 \qquad K > 0$

$$\frac{d^2\psi}{dx^2} + \frac{2m\alpha}{\hbar^2}\,\delta(x)\,\psi = K^2\psi$$

$\underline{x < 0}$
$$\frac{d^2\psi}{dx^2} = K^2\psi$$

$$\psi(x) = A\,e^{-Kx} + B\,e^{+Kx}$$

$$x \to -\infty \qquad \psi(-\infty) = 0 \qquad \Rightarrow \underline{A = 0}$$

$x < 0: \quad \psi(x) = B\, e^{Kx}$

$$= B\, e^{-K|x|}$$

$\underline{x > 0}$

$$\psi(x) = F\, e^{-Kx} + G\, e^{Kx}$$

$$\psi(+\infty) = 0 \implies \underline{G = 0}$$

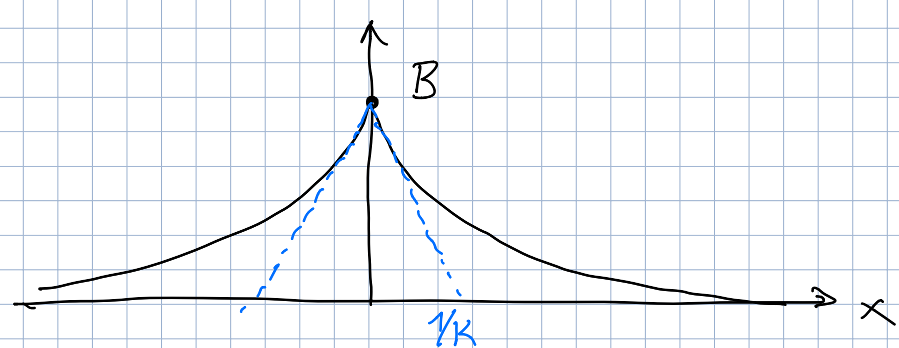
$x > 0: \quad \psi(x) = F\, e^{-Kx}$

$$= F\, e^{-K|x|}$$

$$\psi(x \to 0^+) = \psi(x \to 0^-)$$

$$F = B$$

$\forall x \quad \underline{\psi(x) = B\, e^{-K|x|}}$



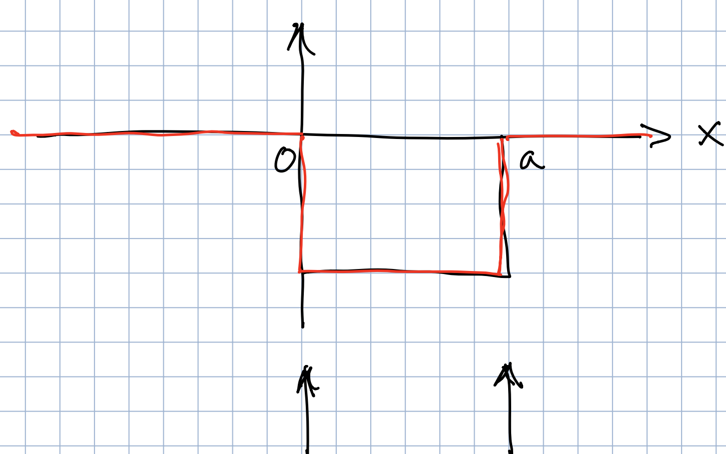
$$\int_{-\infty}^{+\infty} dx \; |\psi(x)|^2 = 1$$

$$= 2B^2 \underbrace{\int_0^{\infty} dx \; e^{-2Kx}}_{\frac{1}{(-2K)} e^{-2Kx} \Big|_0^{\infty}} = 1$$

$$= 2B^2 \frac{1}{2K} = 1$$

$$\underline{B = \sqrt{K}}$$

$$\underline{\forall x}: \; \psi(x) = \sqrt{K}\, e^{-K|x|}$$



$$\psi(0^+) = \psi(0^-)$$

$$\frac{d\psi}{dx}(0^+) = \frac{d\psi}{dx}(0^-)$$

$$\Downarrow$$

FÜR ENDLICHE POT.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\int_{-\varepsilon}^{\varepsilon}$$

$\varepsilon \to 0$

$-\varepsilon \Rightarrow 0^-$

$+\varepsilon \Leftrightarrow 0^+$

$$-\frac{\hbar^2}{2m}\left(\left.\frac{d\psi}{dx}\right|_{x=+\varepsilon} - \left.\frac{d\psi}{dx}\right|_{x=-\varepsilon}\right)$$

$$+ \int_{-\varepsilon}^{\varepsilon} dx\; V(x)\,\psi(x) = E \underbrace{\int_{-\varepsilon}^{\varepsilon} dx\;\psi(x)}_{\langle\psi\rangle 2\varepsilon}$$

$$\xrightarrow[\varepsilon \to 0]{} 0$$

1) $V(x)$ ENDLICH IM INTERVAL $[-\varepsilon, \varepsilon]$

$$\int_{-\varepsilon}^{\varepsilon} dx \; V(x)\psi(x) = \langle V\psi \rangle \, 2\varepsilon \longrightarrow 0$$

$$\left.\frac{d\psi}{dx}\right|_{x=+\varepsilon} = \left.\frac{d\psi}{dx}\right|_{x=-\varepsilon}$$

2) $V(x)$ UNENDLICH IM INTERVAL $[-\varepsilon, \varepsilon]$

$$V(x) = -\alpha\,\delta(x)$$

$$\int_{-\varepsilon}^{\varepsilon} dx \; V(x)\psi(x) = -\alpha\,\psi(0)$$

$$\underbrace{\left.\frac{d\psi}{dx}\right|_{0^+} - \left.\frac{d\psi}{dx}\right|_{0^-}}_{\Delta\left(\frac{d\psi}{dx}\right)_{x=0}} = -\frac{2m}{\hbar^2}\,\alpha\,\psi(0)$$

$$\psi(x) = \sqrt{\kappa}\, e^{-\kappa |x|}$$

$$= \begin{cases} \sqrt{\kappa}\, e^{\kappa x} & x < 0 \\ \sqrt{\kappa}\, e^{-\kappa x} & x \geq 0 \end{cases}$$

$$\left.\frac{d\psi}{dx}\right|_{0^+} = -\kappa^{3/2}$$

$$\left.\frac{d\psi}{dx}\right|_{0^-} = \kappa^{3/2}$$

$$\Delta\left(\frac{d\psi}{dx}\right)_{x=0} = -2\kappa^{3/2}$$

$$-2\kappa^{3/2} = -\frac{2m}{\hbar^2}\,\alpha\,\kappa^{1/2}$$

$$\kappa = \frac{m}{\hbar^2}\alpha$$ 1 LÖSUNG

$$E = -\frac{\hbar^2}{2m}\kappa^2 = -\frac{m\alpha^2}{2\hbar^2}$$

1 GEBUNDENER ZUSTAND

- $\underline{\underline{E > 0}}$ $\qquad E = \frac{\hbar^2}{2m} k^2$

$$\frac{d^2\psi}{dx^2} + \frac{2m\alpha}{\hbar^2}\,\delta(x)\,\psi = -k^2\psi$$

$\underline{x < 0}$ $\qquad \dfrac{d^2\psi}{dx^2} = -k^2\psi$

$$\rightsquigarrow \quad \psi(x) = A\,e^{ikx} + B\,e^{-ikx}$$

$\underline{x > 0}$

$$\rightsquigarrow \quad \psi(x) = F\,e^{ikx} + G\,e^{-ikx}$$

1°) $\psi(0^-) = \psi(0^+)$

$$A + B = F + G$$

2°)

$$\underbrace{\left.\frac{d\psi}{dx}\right|_{0^+} - \left.\frac{d\psi}{dx}\right|_{0^-}} = -\frac{2m}{\hbar^2}\,\alpha\,\psi(0)$$

$$\left.\frac{d\psi}{dx}\right|_{0^+} = ik\,(F - G)$$

$$\left.\frac{d\psi}{dx}\right|_{0^-} = ik\,(A - B)$$

$$ik\,(F - G - A + B) = -\frac{2m}{\hbar^2}\,\alpha\,(A + B)$$

$$(F - G - A + B) = +\frac{2m\,i\alpha}{\hbar^2 k}\,(A + B)$$

$$\frac{m\alpha}{\hbar^2 k} \equiv \beta$$

$$= 2i\beta\,(A + B)$$

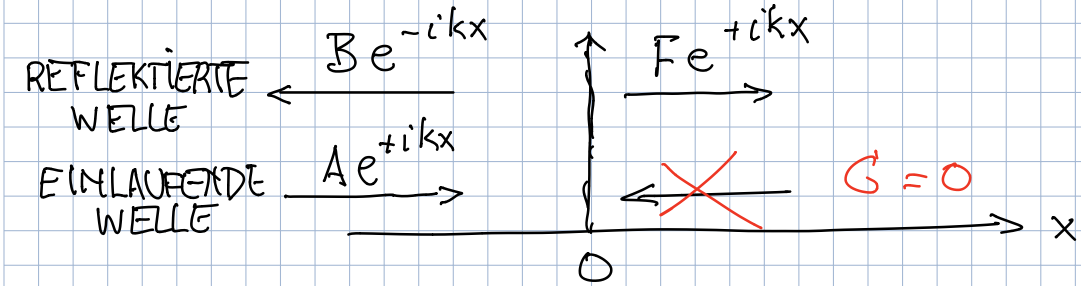
$$F - G = A\,(1 + 2i\beta) - B\,(1 - 2i\beta)$$

F, G, A, B 2 BEZIEHUNGEN

$$e^{ikx} \quad e^{-\frac{i}{\hbar}Et} \longrightarrow$$

$$e^{-ikx} \quad e^{-\frac{i}{\hbar}Et} \longleftarrow$$

RANDBEDINGUNG (EINLAUFENDE WELLE)



$$F = A + B$$

$$F = A(1 + 2i\beta) - B(1 - 2i\beta)$$

$$\cancel{A} + B = A(\cancel{1} + 2i\beta) - B(1 - 2i\beta)$$

$$2B(1 - i\beta) = 2i\beta A$$

$$\boxed{B = \frac{i\beta}{1 - i\beta} A}$$

$$\beta \sim \frac{1}{k} \sim \frac{1}{\sqrt{E}}$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$E \gg \qquad \beta \to 0 \qquad B = 0$

$E \ll \qquad \beta \to \infty \qquad B = -A$

PERFEKTE REFLEKTION

$$F = A + B = \left(1 + \frac{i\beta}{1 - i\beta}\right) A$$

$$F = \frac{1}{1 - i\beta} A$$

$E \gg \qquad \beta \to 0 \qquad F = A$

PERFEKTE TRANSMISSION

$E \ll \qquad \beta \to \infty \qquad F = 0$

↳ WAHRSCHEINLICHKEIT

$$R = \frac{|B|^2}{|A|^2} = \left|\frac{i\beta}{1 - i\beta}\right|^2 = \frac{\beta^2}{1 + \beta^2}$$

$\Downarrow$

REFLEKTIONS KOEFFICIENT

$$T = \frac{|F|^2}{|A|^2} = \left| \frac{1}{1-i\beta} \right|^2 = \frac{1}{1+\beta^2}$$

$\Downarrow$

TRANSMISSIONSKOEFFICIEN

$$\underline{\underline{R + T = 1}}$$

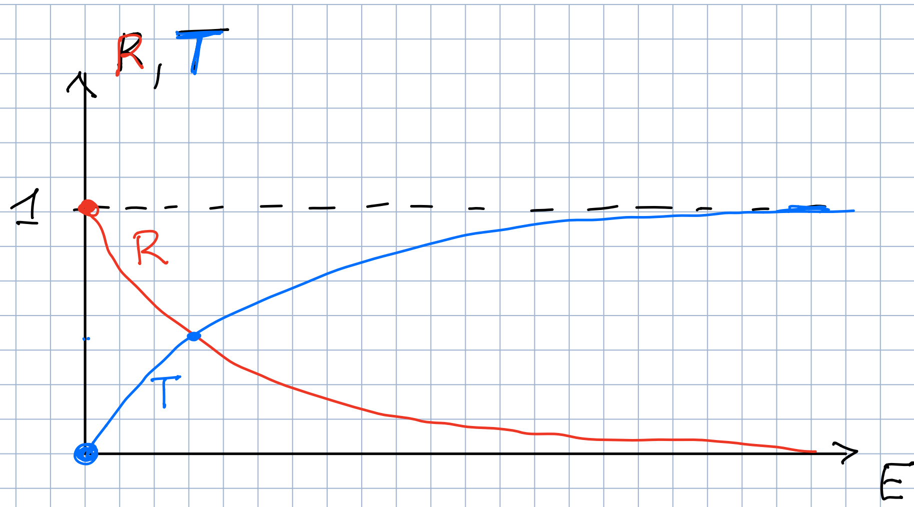
!

$$\frac{m\alpha}{\hbar^2 k} \equiv \beta \qquad E = \frac{\hbar^2 k^2}{2m}$$

$$\beta^2 = \frac{m^2 \alpha^2}{\hbar^4 k^2} \doteq \frac{m \alpha^2}{2\hbar^2 E}$$

$$R = \frac{\beta^2}{1+\beta^2} = \frac{1}{1 + 1/\beta^2} = \frac{1}{1 + \frac{2\hbar^2 E}{m\alpha^2}}$$

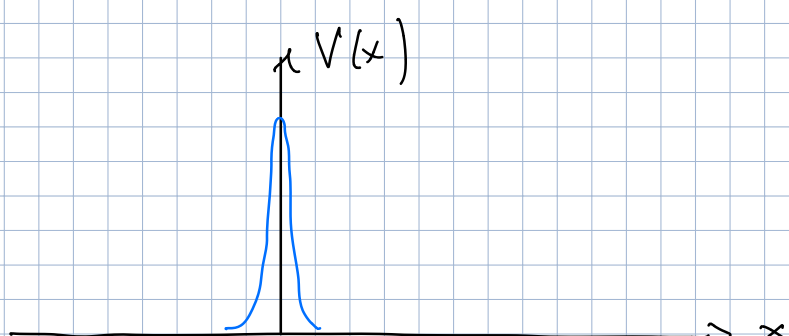
$$T = \frac{1}{1+\beta^2} = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}}$$



↳ WELLENPAKET AUS EBENE WELLEN

$$V(x) = -\alpha\,\delta(x) \qquad \alpha > 0$$

? $$V(x) = +\alpha\,\delta(x) \qquad \alpha > 0$$



$E < 0$: $\alpha \to -\alpha$ KEINE LÖSUNG!

$E > 0$: $\alpha \to -\alpha$ SELBE STREUZUSTÄNDE

R, T FUNKTION VON α^2