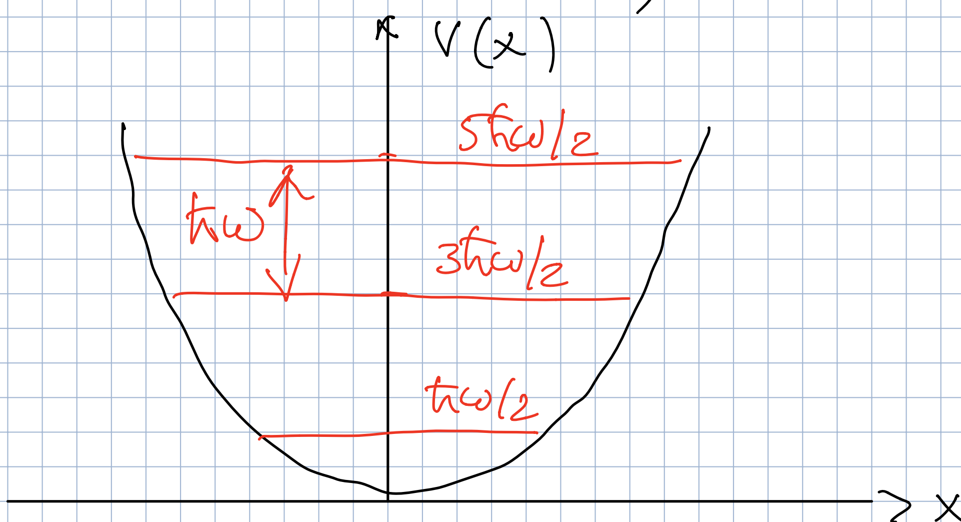
$\Rightarrow$ VORLESUNG 5 QM

2.3 HARMONISCHER OSZILLATOR

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \underbrace{\frac{1}{2}m\omega^2x^2}_{V(x)}\psi = E\psi$$



$$\rightsquigarrow \psi_0 \curvearrowright E_0$$

$$a_+\psi_0 \curvearrowright E_0 + \hbar\omega$$

$$a_+^n\psi_0 \curvearrowright E_0 + n\hbar\omega$$

$$a_- \psi_0 = 0$$

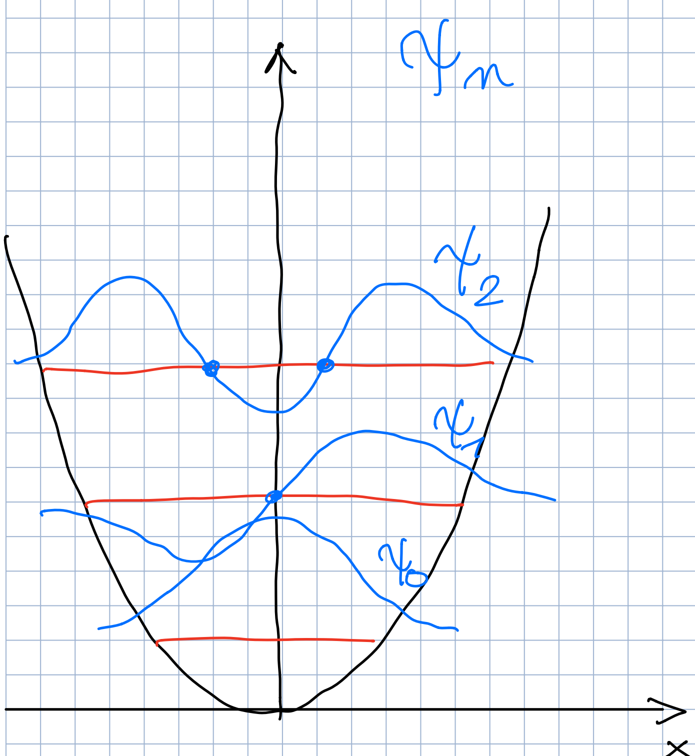
$$\Downarrow$$

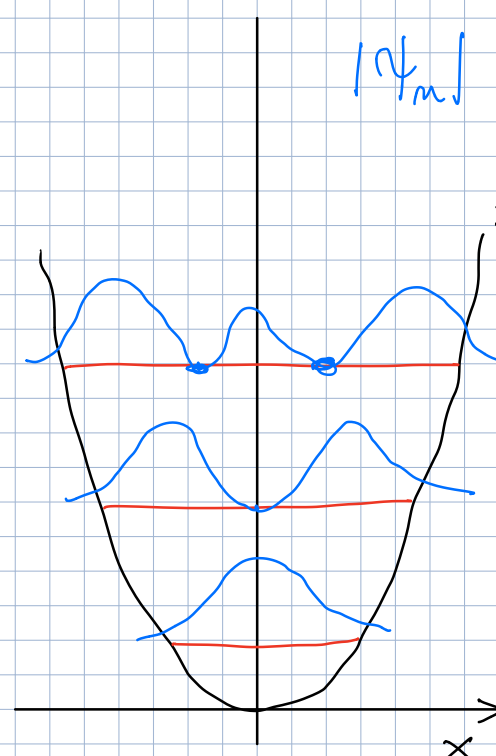
$$\psi_0(x) = C\, e^{-\frac{m\omega}{2\hbar} x^2}$$

$$E_0 = \frac{\hbar\omega}{2}$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$n = 0, 1, 2, \ldots$





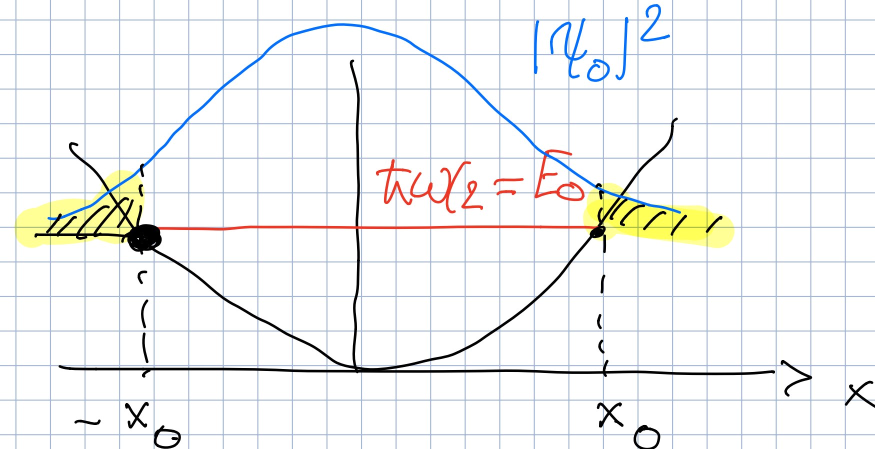
$$\psi_0 \sim e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_1 \sim x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_2 \sim (x^2 + c)\, e^{-\frac{m\omega}{2\hbar}x^2}$$

n GERADE $\longrightarrow \psi(-x) = \psi(x)$

n UNGERADE $\longrightarrow \psi(-x) = -\psi(x)$



$$\longrightarrow \quad V(x_0) = \frac{1}{2} m\omega^2 x_0^2$$

$$= E_0 = \frac{\hbar\omega}{2}$$

$$\Downarrow$$

$$x_0 = \left(\frac{\hbar}{m\omega}\right)^{1/2}.$$

$\longrightarrow$ n ZUSTAND

$$V(x_n) = \frac{1}{2} m\omega^2 x_n^2$$

$$= E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$x_n = \left(\frac{2E_n}{m\omega^2}\right)^{1/2}$$

$$P_0 = 2\int_{x_0}^{\infty} dx \, |\psi_0(x)|^2$$

$$= 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{x_0}^{\infty} dx$$

$\uparrow$

$$|\psi_0|^2 = C^2 \, e^{-\frac{m\omega}{\hbar}x^2}$$

$$= C^2 \, e^{-(x/x_0)^2}$$

$$P_0 = \frac{2}{\sqrt{\pi}} \, \frac{1}{x_0} \int_{x_0}^{\infty} dx \; e^{-(x/x_0)^2}$$

$$\xi = x/x_0$$

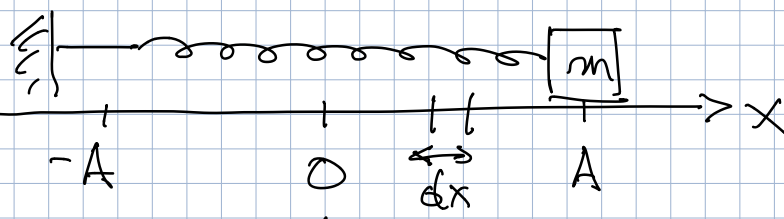
$$P_0 = \frac{2}{\sqrt{\pi}} \int_{1}^{\infty} d\xi \; e^{-\xi^2}$$

$$= 0.157$$

15.7 % !

↳ WAHRSCHEINLICHKEITEN

KL ⟷ QM OSZILLATOR

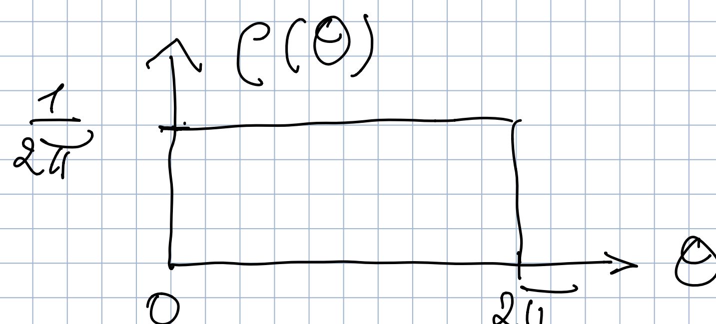
- 

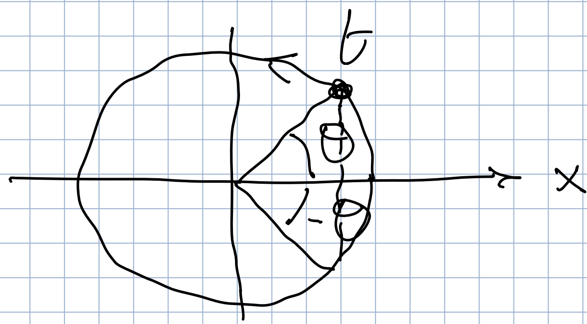
$$x(t) = A\cos(\omega t)$$

$$\rightsquigarrow \; P_{KL}(x)\, dx$$

$$\Rightarrow \; t \gg T = \frac{2\pi}{\omega}$$

$$\Theta \equiv \omega t$$





$$\longrightarrow \quad P_{kl}(x)\,dx = 2\,\rho(\Theta)\,d\Theta$$

$$= \frac{2}{2\pi}\,d\Theta$$

$$x = A\cos\Theta$$

$$\left|\frac{dx}{d\Theta}\right| = |A\sin\Theta|$$

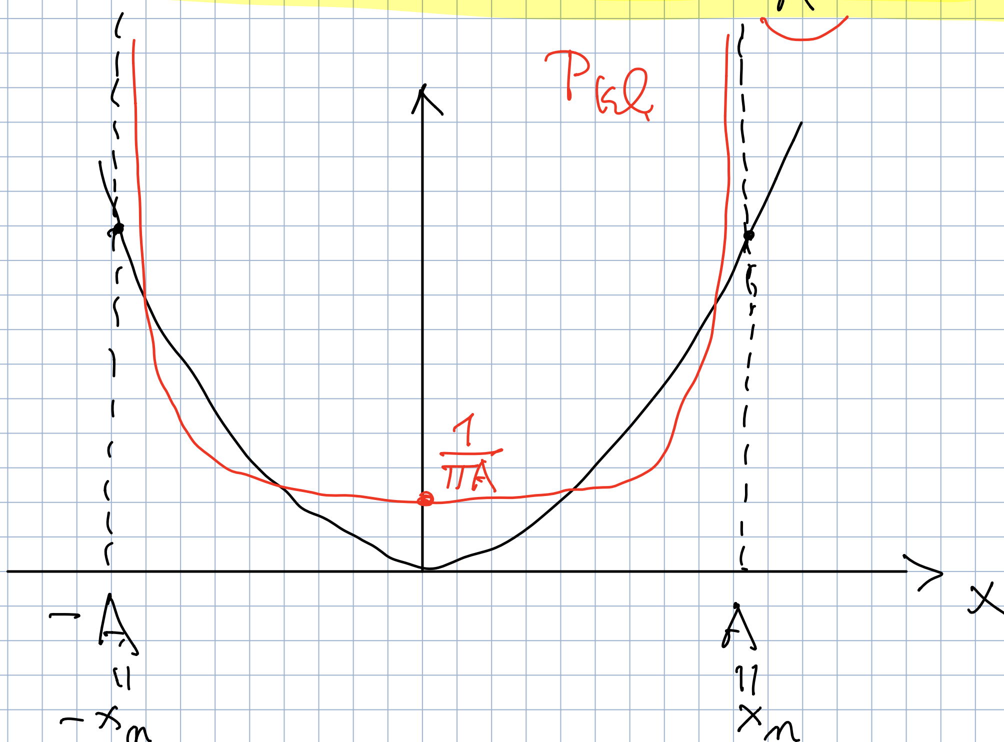
$$= A\left(1-\frac{x^2}{A^2}\right)^{1/2}$$

$$P_{kl}(x) = \frac{2}{2\pi}\left|\frac{d\Theta}{dx}\right| \qquad \Theta = \omega t$$

$$= \frac{1}{\pi}\,\omega\left|\frac{dt}{dx}\right|$$

$$= \frac{\omega}{\pi}\,\frac{1}{|v|}$$

$$P_{Kl}(x) = \frac{1}{\pi} \frac{1}{A\left(1-\frac{x^2}{A^2}\right)^{1/2}}$$



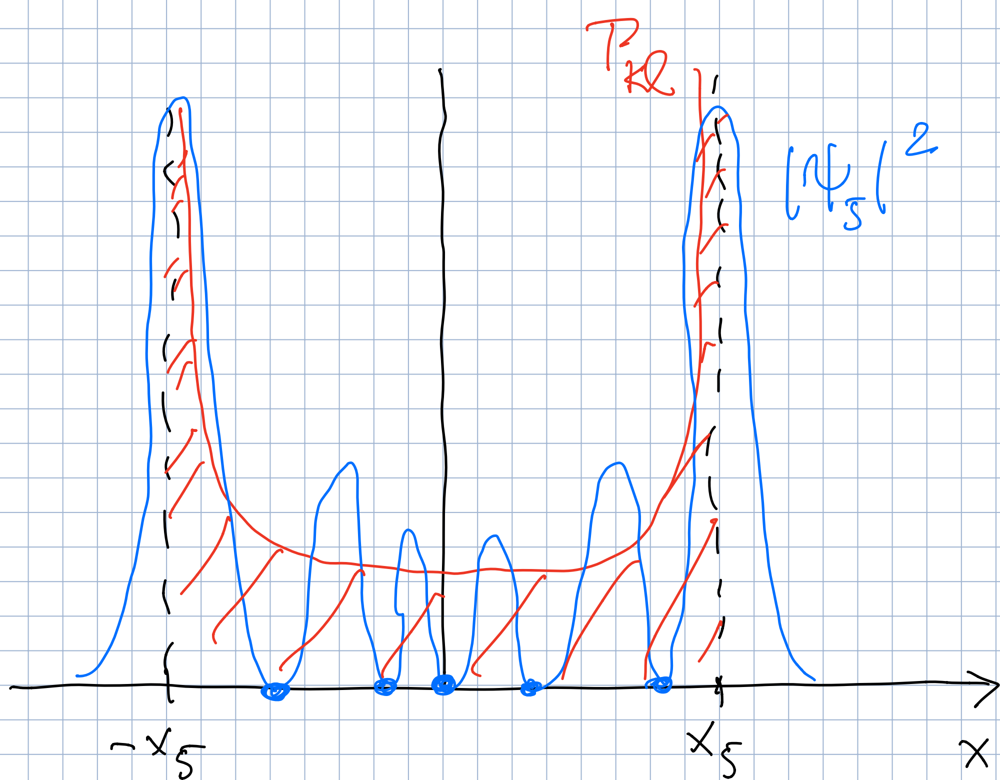
$$P_{QM_n}(x) = |\psi_n(x)|^2$$

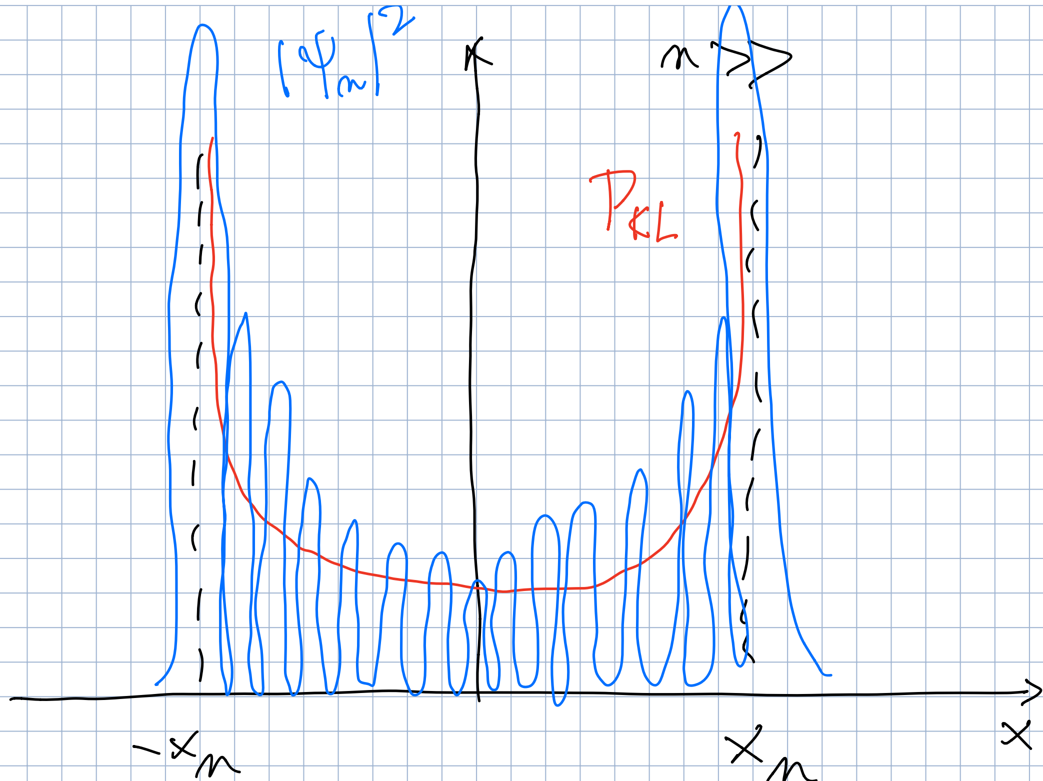
$$x_n = \left(\frac{2E_n}{m\omega^2}\right)^{1/2} \qquad E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

$$= \left(\frac{\hbar}{m\omega}\right)^{1/2} \sqrt{2n+1}$$

$$P_{kl}(x) = \frac{1}{\pi \left(\frac{\hbar}{m\omega}\right)^{1/2} \sqrt{2n+1}}$$

$$\cdot \frac{1}{\left(1 - \frac{x^2}{(\hbar/m\omega)} \frac{1}{2n+1}\right)^{1/2}}$$





n KNOTEN

$n \uparrow$

KORRESPONDENZ

KL $\longleftrightarrow$ QM

$n \to \infty$

2.4 FREIE TEILCHEN

$$V(x) = 0 \qquad \forall x$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\Big\downarrow \quad E = \frac{\hbar^2 k^2}{2m} \quad \Longleftarrow$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

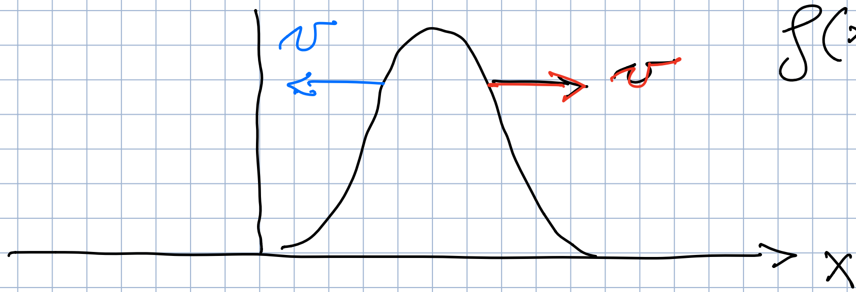
$$\hookrightarrow \quad \psi_k(x) = A e^{-ikx} + B e^{+ikx}$$

$$\forall k \in \mathbb{R}$$

$$\rightsquigarrow \quad \Psi_k(x,t) = A e^{-\frac{i}{\hbar}Et} e^{-ikx} + B e^{-\frac{i}{\hbar}Et} e^{+ikx}$$

$$= A e^{-ik\left(x + \frac{\hbar k}{2m}t\right)} + B e^{+ik\left(x - \frac{\hbar k}{2m}t\right)}$$

$$f(x \mp v t)$$



$f(x - vt)$

GESCHWINDIGKEIT

$$E = \frac{\hbar^2 k^2}{2m}$$

$$v_{QM} = \frac{\hbar k}{2m} = \sqrt{\frac{E}{2m}}$$

IMPULS $p = \hbar k$

KL. $E = \frac{1}{2} m v_{KL}^2 \Rightarrow v_{KL} = \sqrt{\frac{2E}{m}}$

$$v_{QM} = \frac{1}{2} v_{KL}$$

$\Rightarrow$ NORMIERUNG

$$\Psi_k(x,t) = B\, e^{+ik\left(x - \frac{\hbar k}{2m} t\right)}$$

$$1 = \int_{-\infty}^{+\infty} dx \, |\Psi_k(x,t)|^2$$

$$= |B|^2 \int_{-\infty}^{+\infty} dx \ \underbrace{\left| e^{ik(\ldots)} \right|^2}_{1}$$

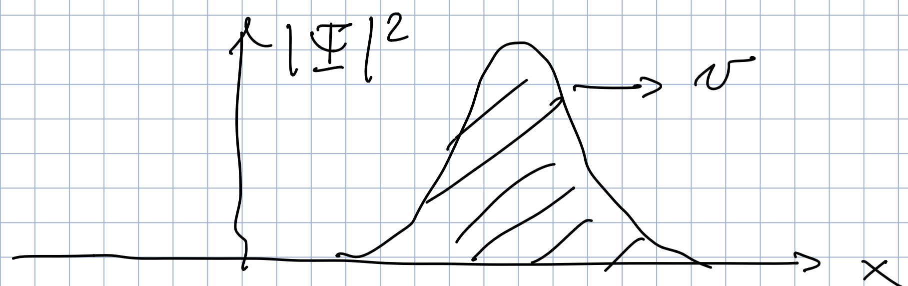
$$\neq \infty$$

$$\Psi(x,t) = \sum_k \underbrace{c_k}_{} \Psi_k(x,t)$$

$$k \in \mathbb{R}$$

$$\Rightarrow \quad \Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \ \phi(k) \ e^{ik\left(x - \frac{\hbar k}{2m} t\right)}$$

WELLENPAKET



$\phi(k)$?

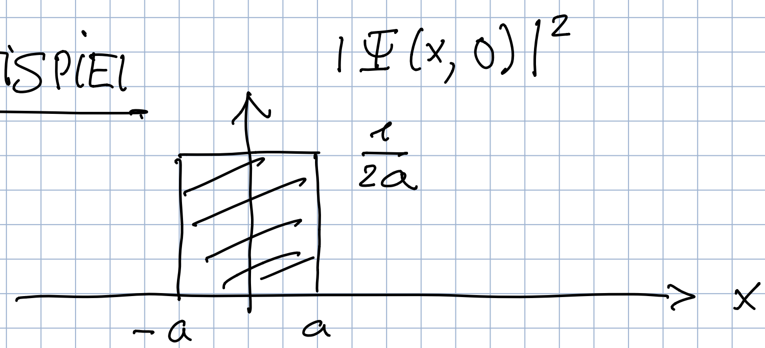
<u>FOURIER TF</u> (FT)

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \, \underbrace{\phi(k)} \, e^{ikx}$$

$$\Big\downarrow \quad \text{FT}$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \, e^{-ikx} \, \Psi(x,0)$$

- <u>BEISPIEL</u>



$$\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{2a}} & -a < x < a \\ 0 & |x| > a \end{cases}$$

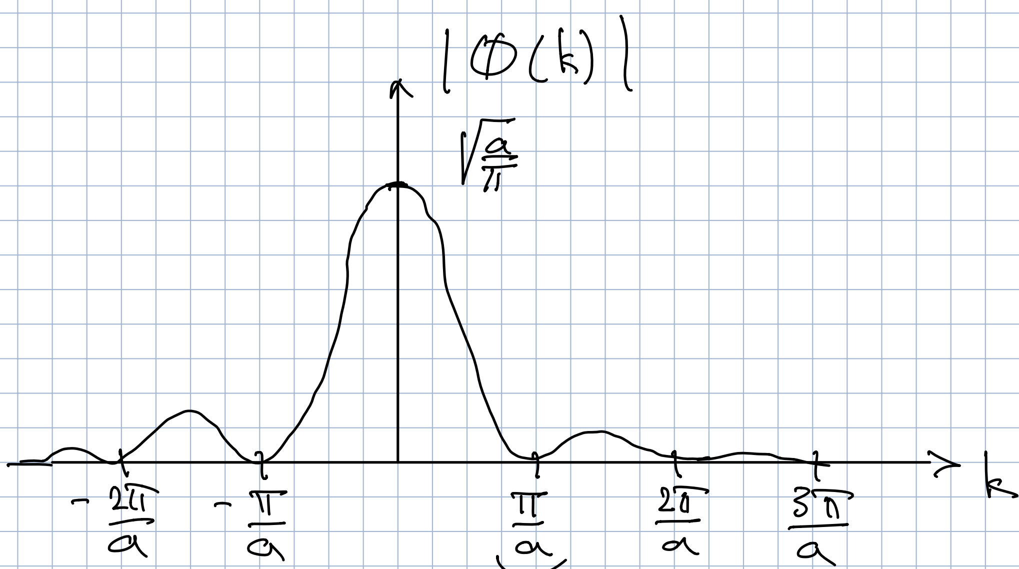
$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dk \; e^{-ikx} \; \frac{1}{\sqrt{2a}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \frac{1}{(-ik)} \; e^{-ikx} \Big|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \left(\frac{1}{-ik} \left(e^{-ika} - e^{ika} \right) \right)$$

$$\frac{2}{k} \sin ka$$

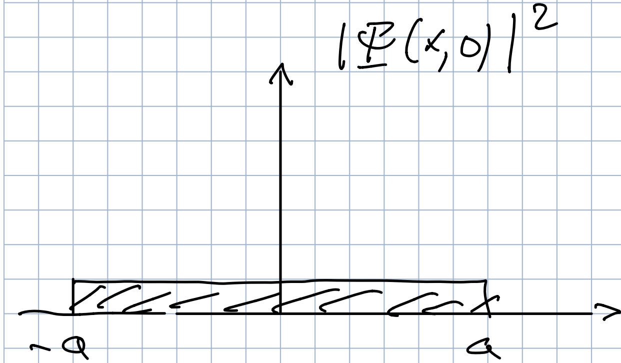
$$\boxed{\phi(k) = \frac{1}{\sqrt{\pi a}} \frac{\sin ka}{k}} = \sqrt{\frac{a}{\pi}} \frac{\sin ka}{ka}$$



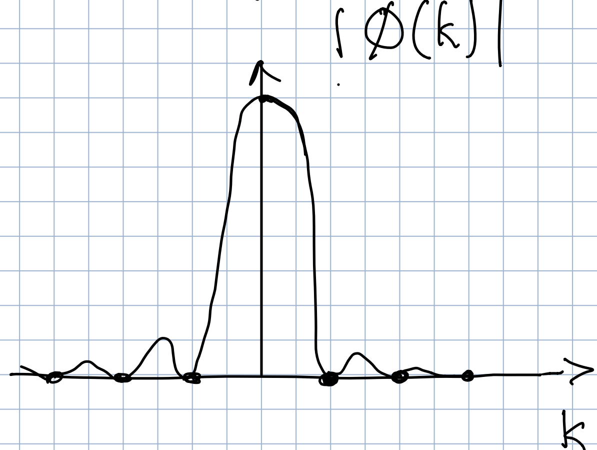
$$ka = \pm \pi, \ \pm 2\pi, \ \dots$$

$$\frac{\sin z}{z} \xrightarrow[z \to 0]{} 1$$

$a \gg$



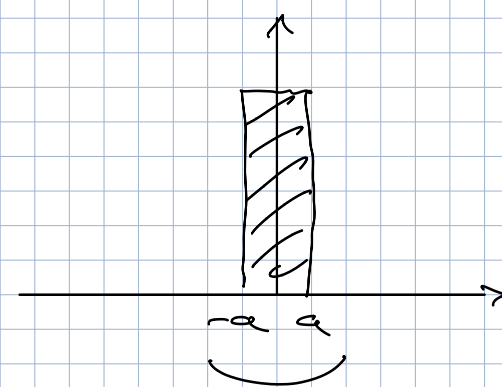
$\Downarrow$



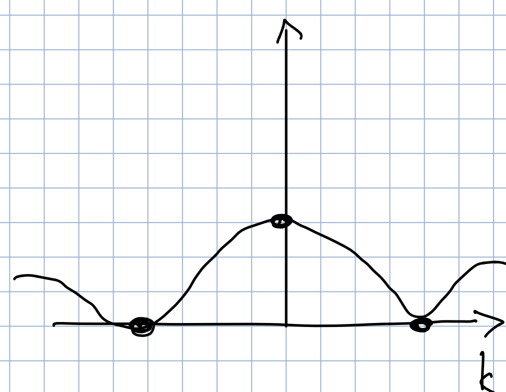
NICHT - LOKALISIERT
IM ORT

LOKALISIERT
IM IMPULS

$a \ll$



$\Downarrow$

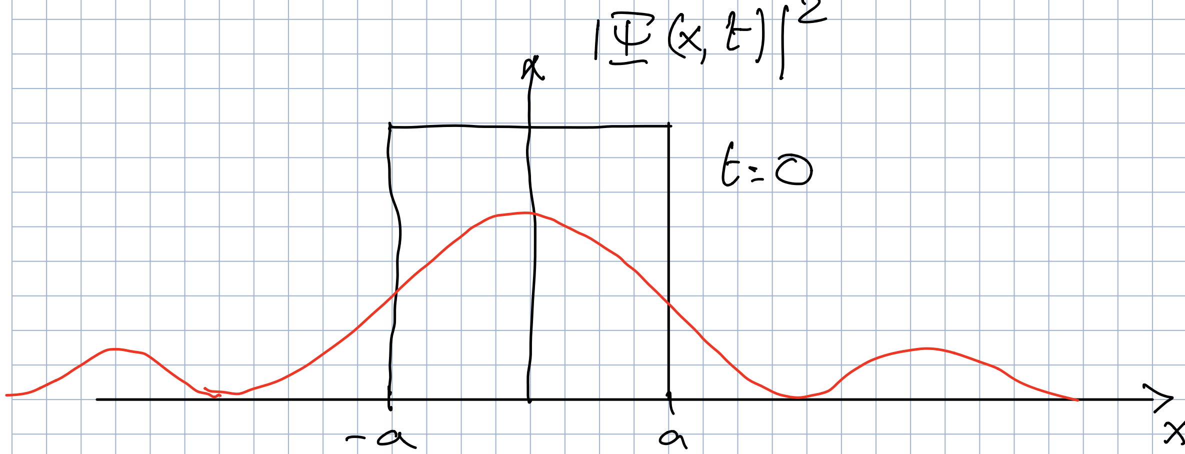


LOKALISIERT
IM **ORT**

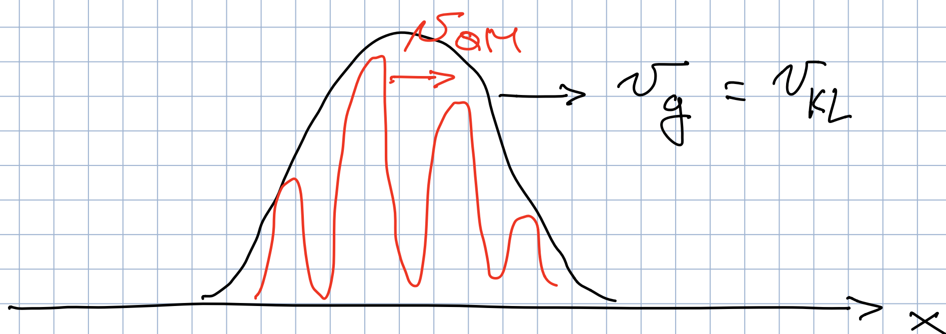
NICHT - LOKALISIERT
IM IMPULS

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \; \phi(k) \, e^{ik\left(x - \underbrace{\frac{\hbar k}{2m}}\, t\right)}$$

$$= \frac{1}{\pi\sqrt{2a}} \int_{-\infty}^{+\infty} dk \left(\frac{\sin ka}{k}\right) e^{ikx} \, e^{-i\frac{\hbar k^2}{2m} t}$$



$$v_{QM} = \frac{\hbar k}{2m}$$



$$v_{QM} = \frac{\omega}{k}$$

$$\frac{d\omega}{dk} = 2\left(\frac{\hbar k}{2m}\right)$$

$$v_g = 2\, v_{QM}$$

$$\|$$

$$v_{KL}$$

$$E = \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

$$\omega(k) = \frac{\hbar}{2m} k^2$$