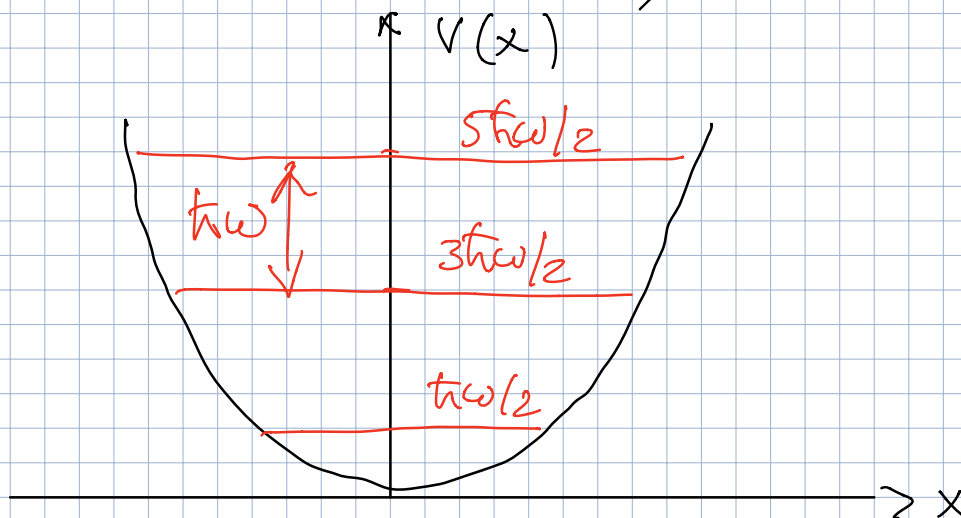


⇒ VORLESUNG 5 QM

2.3 HARMONISCHER OSZILLATOR

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \underbrace{\frac{1}{2} m \omega^2 x^2}_{V(x)} \psi = E \psi$$



$$\psi_0 \rightarrow E_0$$

$$a_+ \psi_0 \rightarrow E_0 + \hbar\omega$$

$$a_+^n \psi_0 \rightarrow E_0 + n \hbar\omega$$

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$$\frac{a - \psi_0 = 0}{}$$



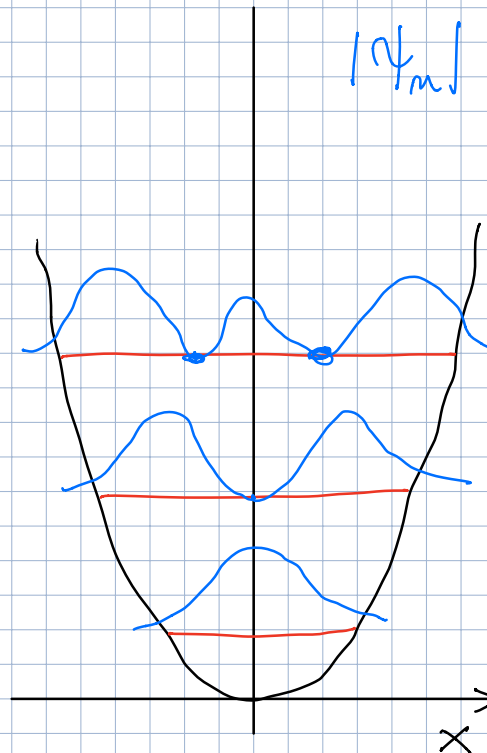
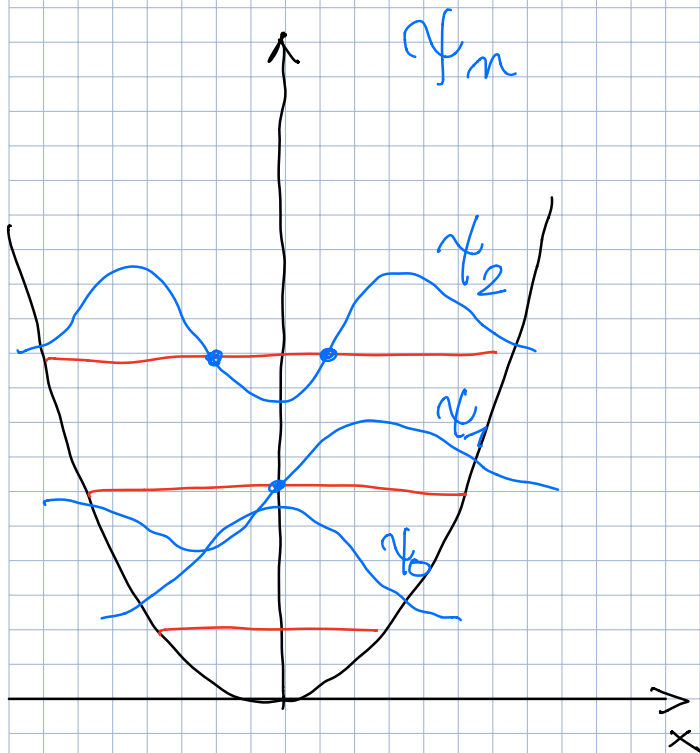
$$\psi_0(x) = C e^{-\frac{m\omega}{2\hbar} x^2}$$

$$E_0 = \frac{\hbar\omega}{2}$$



$$E_n = \hbar\omega \left(n + \frac{1}{2} \right)$$

$$n = 0, 1, 2, \dots$$



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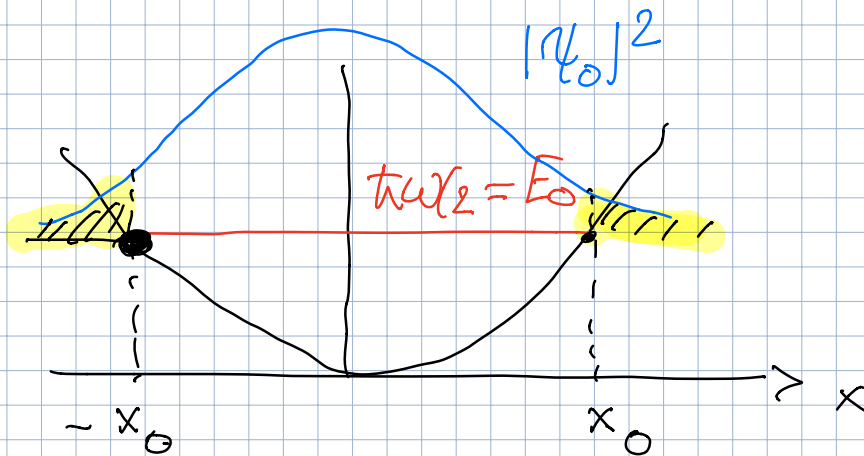
$$\psi_0 \sim e^{-\frac{m\omega}{2\hbar} x^2}$$

$$\psi_1 \sim x e^{-\frac{m\omega}{2\hbar} x^2}$$

$$\psi_2 \sim (x^2 + c) e^{-\frac{m\omega}{2\hbar} x^2}$$

n GERADE $\rightarrow \psi(-x) = \psi(x)$

n UNGERADE $\rightarrow \psi(-x) = -\psi(x)$



$$\begin{aligned} \rightarrow V(x_0) &= \frac{1}{2} m\omega^2 x_0^2 \\ &= E_0 = \frac{\hbar\omega}{2} \\ &\Downarrow \end{aligned}$$

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$$x_0 = \left(\frac{\hbar}{m\omega} \right)^{1/2}$$

→ n ZUSTAND

$$\begin{aligned} V(x_n) &= \frac{1}{2} m\omega^2 x_n^2 \\ &= E_n = \hbar\omega \left(n + \frac{1}{2} \right) \end{aligned}$$

$$x_n = \left(\frac{2E_n}{m\omega^2} \right)^{1/2}$$

$$\begin{aligned} P_0 &= 2 \int_{x_0}^{\infty} dx |N_0(x)|^2 \\ &= 2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \int_{x_0}^{\infty} dx \end{aligned}$$

↑ x_0

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$$|\psi_0|^2 = C^2 e^{-\frac{m\omega}{\hbar} x^2}$$

$$= C^2 e^{-(x/x_0)^2}$$

$$P_0 = \frac{2}{\sqrt{\pi}} \frac{1}{x_0} \int_{x_0}^{\infty} dx e^{-(x/x_0)^2}$$

$$\downarrow \quad \xi = x/x_0$$

$$P_0 = \frac{2}{\sqrt{\pi}} \int_1^{\infty} d\xi e^{-\xi^2}$$

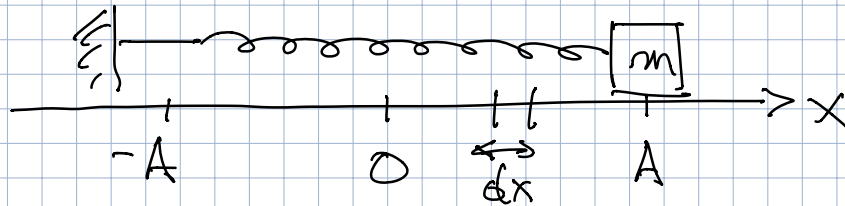
$$= 0.157$$

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L → WAHRSCHEINLICHKEITEN

$\mathcal{K}_L \leftrightarrow \text{QM Oszillator}$

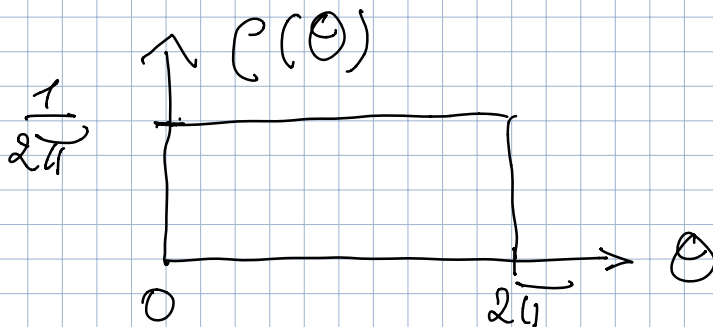


$$x(t) = A \cos(\omega t)$$

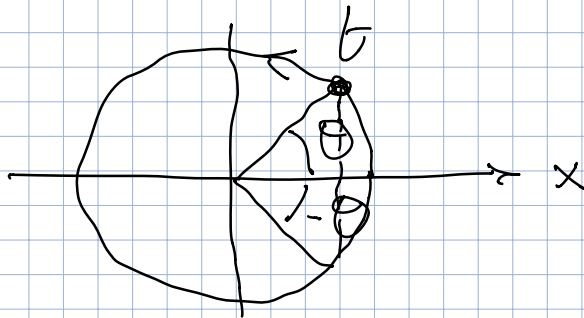
$$\rightsquigarrow P_{\mathcal{K}_L}(x) dx$$

$$\Rightarrow t \gg T = \frac{2\pi}{\omega}$$

$$\theta \equiv \omega t$$



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$$\begin{aligned} \rightarrow P_{kl}(x) dx &= 2 \rho(\theta) d\theta \\ &= \frac{2}{2\pi} d\theta \end{aligned}$$

$$x = A \cos \theta$$

$$\left| \frac{dx}{d\theta} \right| = |A \sin \theta|$$

$$= A \left(1 - \frac{x^2}{A^2} \right)^{1/2}$$

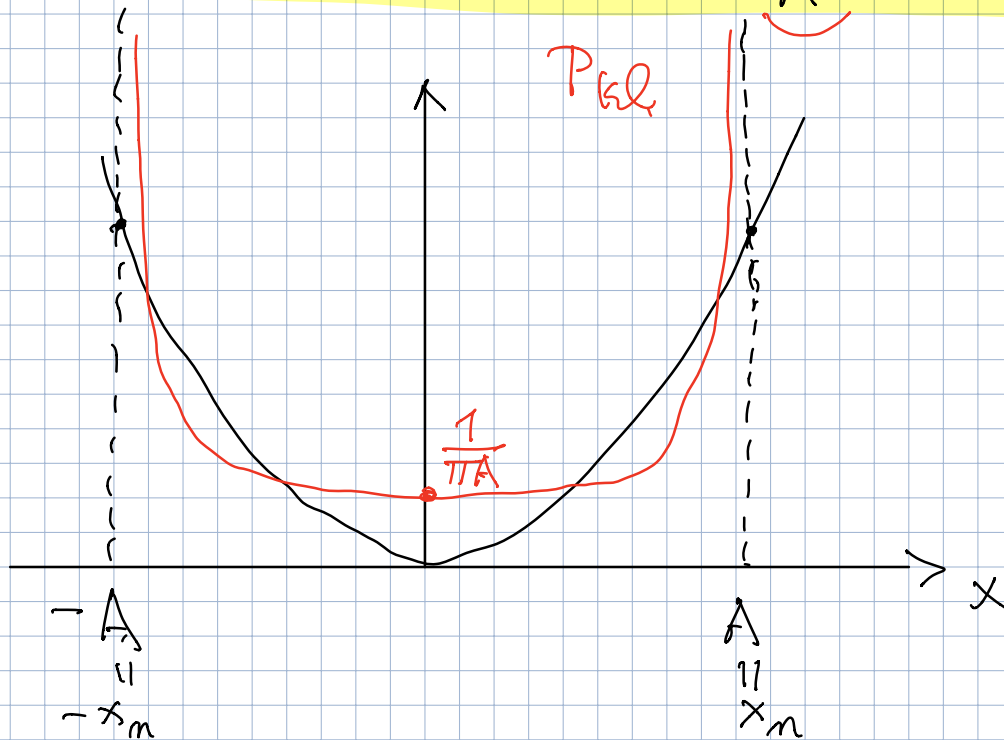
$$P_{kl}(x) = \frac{2}{2\pi} \left| \frac{d\theta}{dx} \right| \quad \theta = \omega t$$

$$= \frac{1}{\pi} \omega \left| \frac{dt}{dx} \right|$$

$$= \frac{\omega}{\pi} \frac{1}{|v|}$$

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$$P_{kl}(x) = \frac{1}{\pi} \frac{1}{A \left(1 - \frac{x^2}{A^2}\right)^{1/2}}$$



$$P_{QM_m}(x) = |\psi_m(x)|^2$$

$$x_m = \left(\frac{2E_m}{m\omega^2} \right)^{1/2}$$

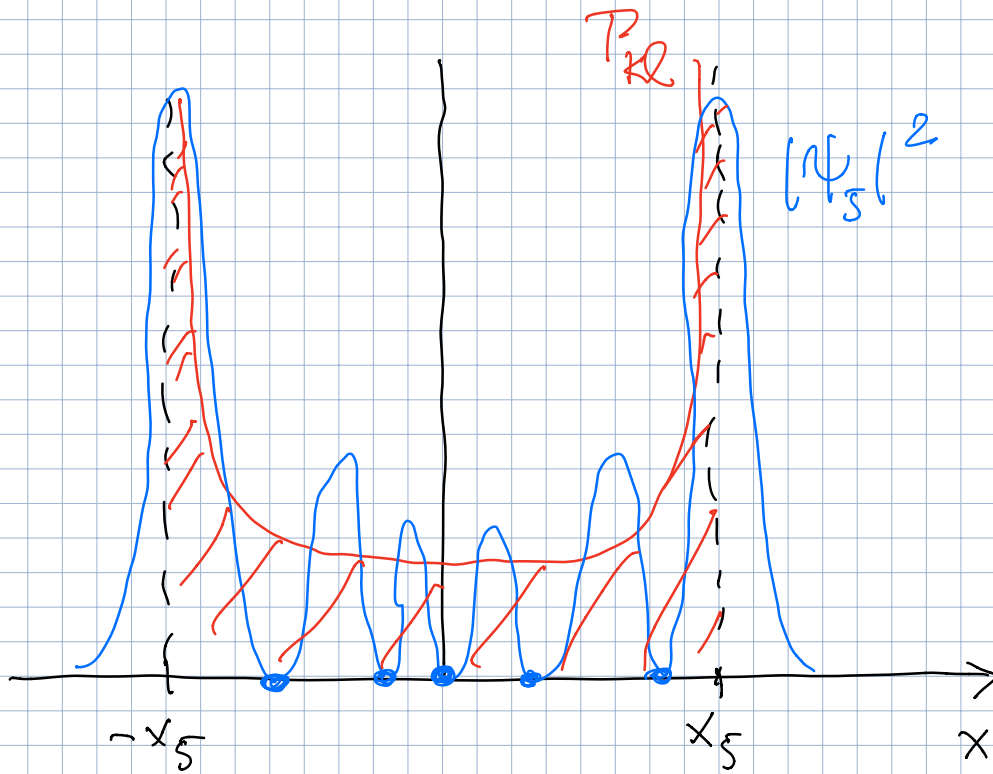
$$E_m = \hbar\omega \left(m + \frac{1}{2} \right)$$

$$= \left(\frac{\hbar}{m\omega} \right)^{1/2} \sqrt{2m+1}$$

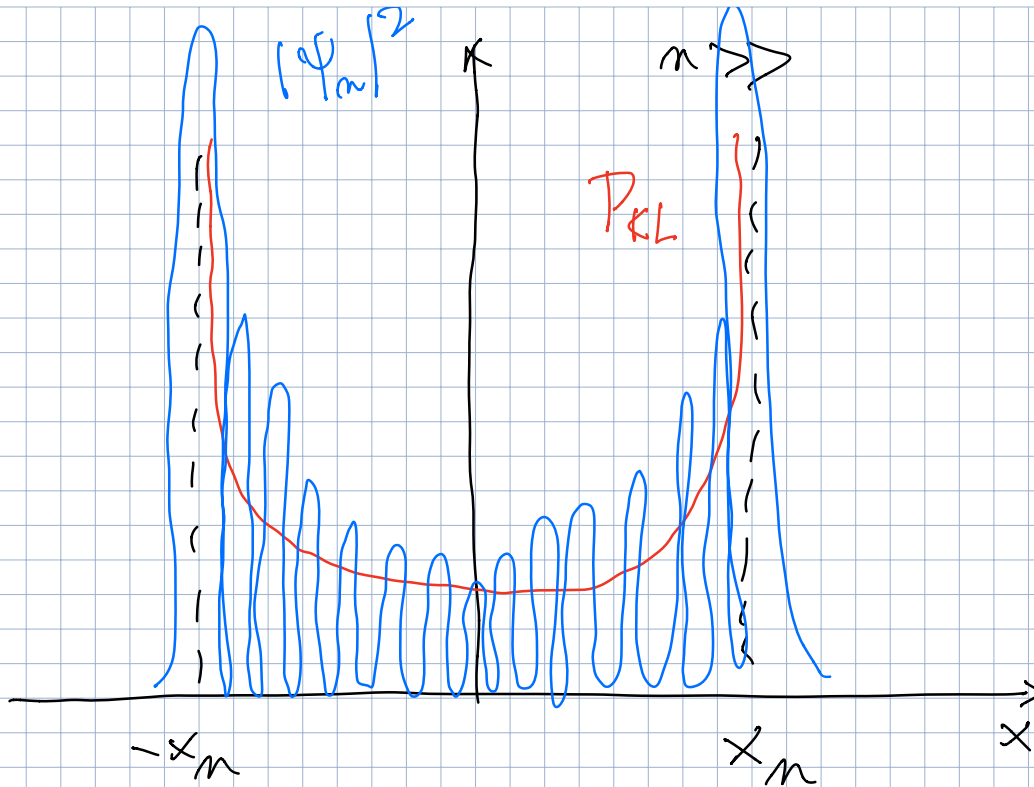
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$$P_{rel}(x) = \frac{1}{\sqrt{\pi} \left(\frac{\hbar}{mc\omega}\right)^{1/2} \sqrt{2n+1}}$$

$$\frac{1}{\left(1 - \frac{x^2}{\left(\frac{\hbar}{mc\omega}\right) \frac{1}{2n+1}}\right)^{1/2}}$$



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n KNOTEN

$n \pi$

KORRESPONDENZ

$KL \leftrightarrow QM$

$n \rightarrow 0$

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2.4 FREIE TEILCHEN

$$V(x) = 0 \quad \forall x$$

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi$$

$$\downarrow E = \frac{\hbar^2 k^2}{2m} \quad \leftarrow$$

$$\frac{d^2 \psi}{dx^2} = -k^2 \psi$$

$$\hookrightarrow \psi_k(x) = A e^{-ikx} + B e^{+ikx}$$

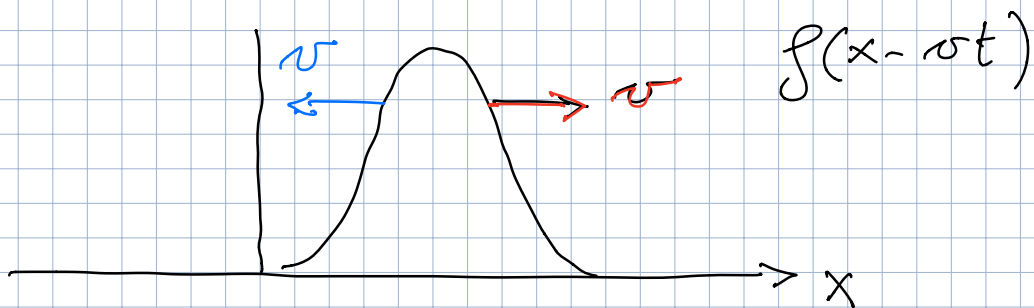
$$\forall k \in \mathbb{R}$$

$$\rightsquigarrow \Psi_k(x, t) = A e^{-\frac{i}{\hbar} E t - ikx} + B e^{-\frac{i}{\hbar} E t + ikx}$$

$$= A e^{-ik \left(x + \frac{\hbar k}{2m} t \right)} + B e^{+ik \left(x - \frac{\hbar k}{2m} t \right)}$$

$$f(x \mp vt)$$

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GESCHWINDIGKEIT $E = \frac{\hbar^2 k^2}{2m}$

$$v_{QM} = \frac{\hbar k}{2m} = \sqrt{\frac{E}{2m}}$$

IMPULS $p = \hbar k$

KL: $E = \frac{1}{2} m v_{KL}^2 \Rightarrow v_{KL} = \sqrt{\frac{2E}{m}}$

$$v_{QM} = \frac{1}{2} v_{KL}$$

⇒ NORMIERUNG

$$\Psi_k(x, t) = B e^{+ik(x - \frac{\hbar k}{2m} t)}$$

$$\textcircled{1} = \int_{-\infty}^{+\infty} dx |\Psi_k(x, t)|^2$$

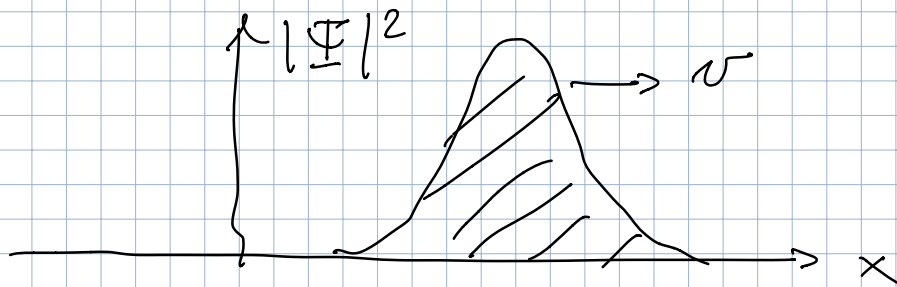
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$$= |B|^2 \int_{-\infty}^{+\infty} dx \underbrace{|e^{ik(\dots)}|^2}_{=1}$$

$\neq \infty$

$$\Psi(x, t) = \sum_{k \in \mathbb{R}} c_k \Psi_k(x, t)$$

$$\Rightarrow \Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \underbrace{\Phi(k)}_{\text{WELLENPAKET}} e^{ik(x - \frac{\hbar k}{2m} t)}$$



$\Phi(k) ?$

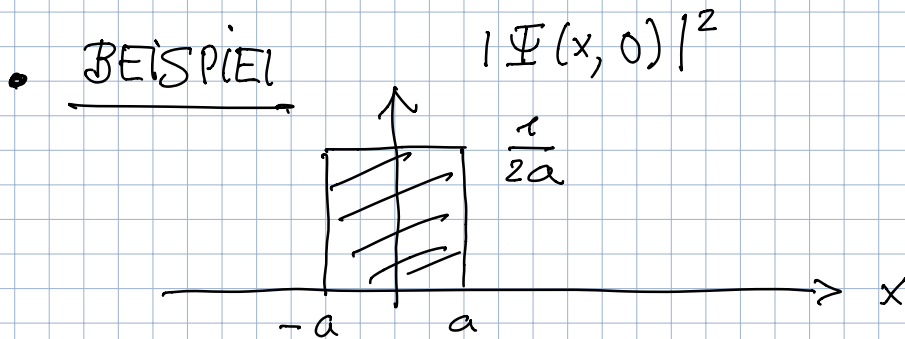
FOURIER TRANSFORM (FT)

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$$\underline{\Psi}(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \underbrace{\Phi(k)} e^{ikx}$$

↓ (FT)

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk e^{-ikx} \underline{\Psi}(x, 0)$$



$$\underline{\Psi}(x, 0) = \begin{cases} \frac{1}{\sqrt{2a}} & -a < x < a \\ 0 & |x| > a \end{cases}$$

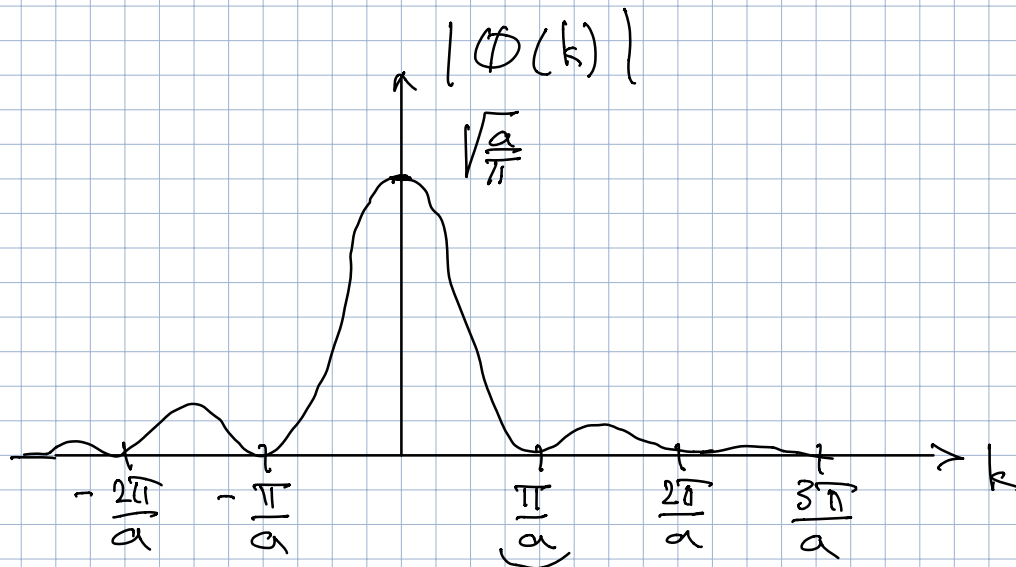
$$\begin{aligned} \Phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dk e^{-ikx} \frac{1}{\sqrt{2a}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \frac{1}{(-ik)} e^{-ikx} \Big|_{-a}^a \end{aligned}$$

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$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \left(\frac{1}{-ik} (e^{-ika} - e^{ika}) \right)$$

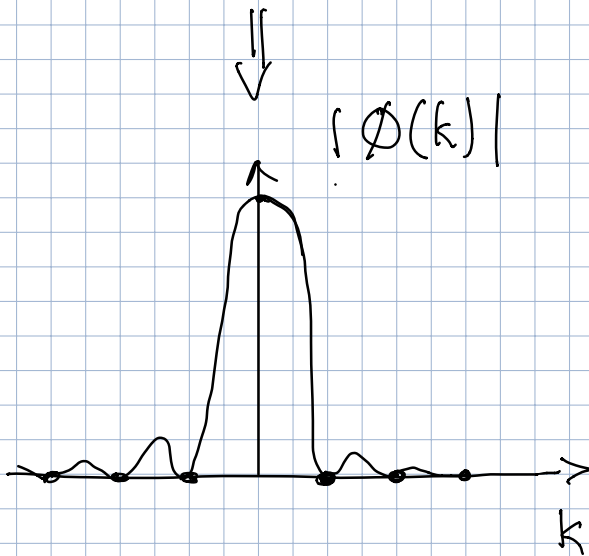
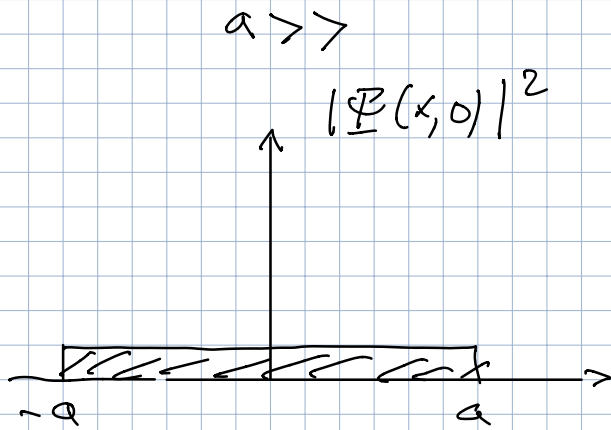
$$\frac{2}{k} \sin ka$$

$$\Phi(k) = \frac{1}{\sqrt{\pi a}} \frac{\sin ka}{k} = \sqrt{\frac{a}{\pi}} \frac{\sin ka}{ka}$$



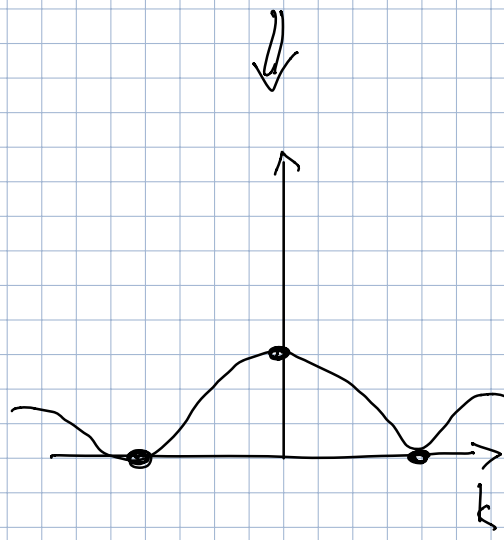
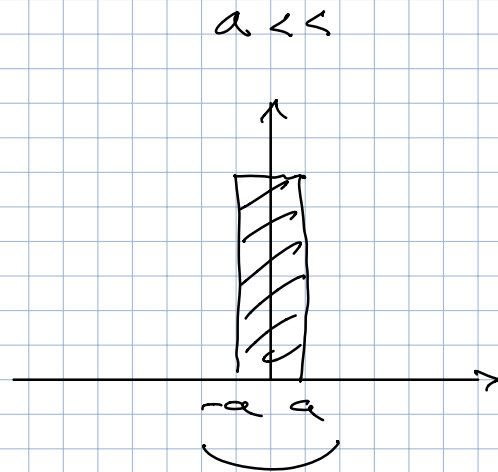
$$ka = \pm \pi, \pm 2\pi, \dots$$

$$\frac{\sin z}{z} \rightarrow 1 \quad z \rightarrow 0$$



NICHT-LOKALISIERT
IM ORT

LOKALISIERT
IM IMPULS



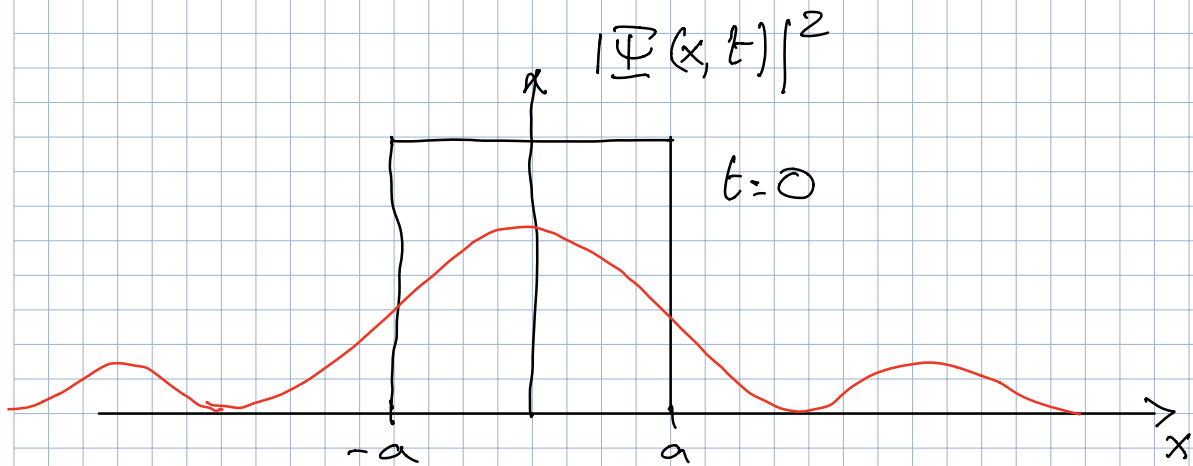
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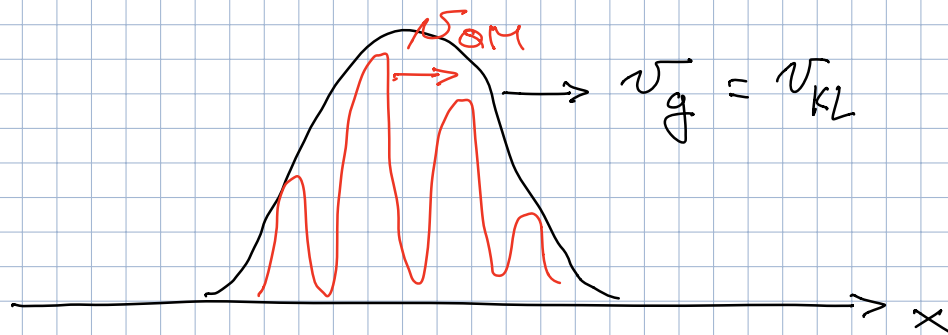
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$$\underline{\Psi}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \phi(k) e^{ik(x - \frac{\hbar k}{2m} t)}$$

$$= \frac{1}{\pi \sqrt{2a}} \int_{-\infty}^{+\infty} dk \left(\frac{\sin ka}{k} \right) e^{ikx} e^{-i \frac{\hbar k^2}{2m} t}$$



$$v_{QM} = \frac{\hbar k}{2m}$$



$$\rightarrow v_{QM} = \frac{\omega}{k}$$

$$\rightarrow \frac{d\omega}{dk} = 2 \left(\frac{\hbar k}{2m} \right)$$

$$v_g = 2 v_{QM}$$

||

$$v_{KL}$$

$$E = \hbar \omega \\ = \frac{\hbar^2 k^2}{2m}$$

$$\omega(k) = \frac{\hbar}{2m} k^2$$