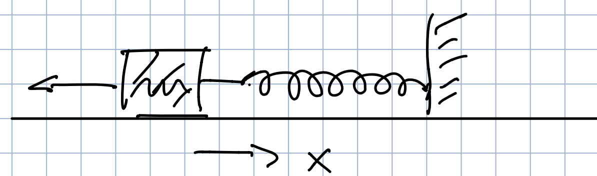
⇒ VORLESUNG 4 QM

↳ 2.3 HARMONISCHE OSCILLATOR

- **KLASSISCHE H.O.**

↳



$$F = -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

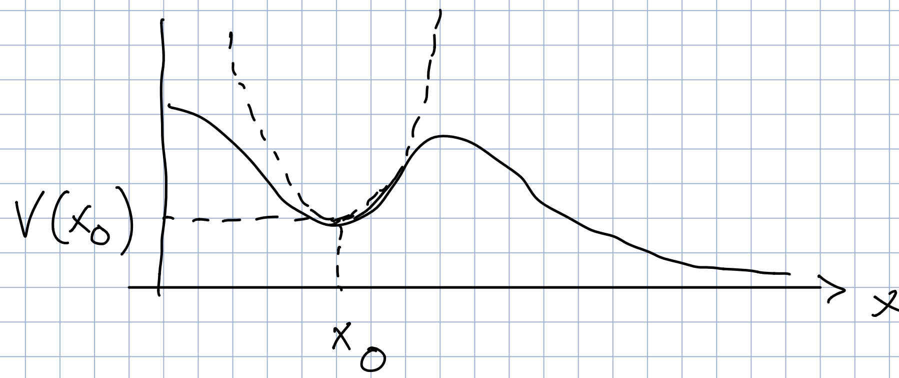
$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$F = -\frac{dV}{dx}$$

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

↳



$$V(x) = V(x_0) + \cancel{V'(x_0)}(x - x_0)$$

$$+ \frac{1}{2} V''(x_0)(x - x_0)^2$$

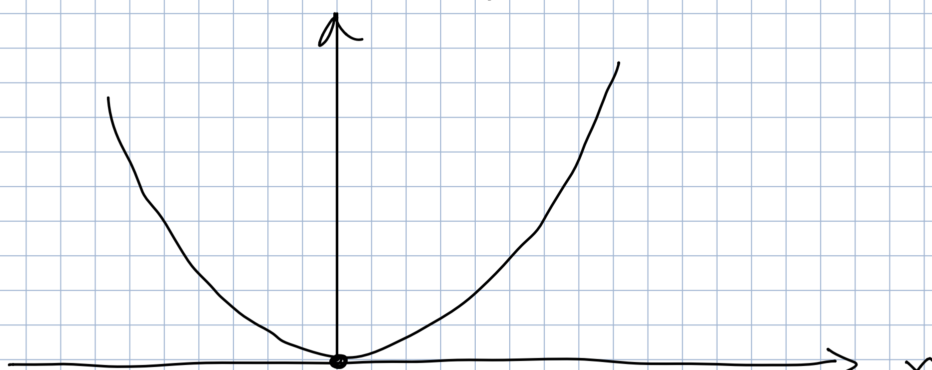
$$+ \ldots$$

$$V(x) - V(x_0) = \frac{1}{2} \underbrace{V''(x_0)}_{k} (x - x_0)^2$$

$$+ \ldots$$

- QUANTUM H.O.

$$V(x) = \frac{1}{2} m\omega^2 x^2$$



$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\chi$$

REIHEN ANSATZ

ALGEBRAISCHE METHODE

$$\hat{p} = -i\hbar \frac{d}{dx}$$

$$\frac{\hat{p}^2}{2m}\psi + \frac{1}{2} m\omega^2 x^2 \psi = E\psi$$

$$\Rightarrow \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$\hat{H} \psi = E \psi$$

$$(i\hat{p} + m\omega x) \cdot (-i\hat{p} + m\omega x)$$

$$= \hat{p}^2 + i m\omega (\hat{p} x - x \hat{p}) + m^2 \omega^2 x^2$$

$$= 2m\hat{H} + i m \omega \underbrace{(\hat{p} x - x\hat{p})}_{-i\hbar}$$

$$x^2 - a^2 = (x-a)(x+a)$$

$$x^2 + b^2 = x^2 - (ib)^2$$

$$= (x - ib)(x + ib)$$

$$x\hat{p} - \hat{p}x = [x, \hat{p}] \stackrel{?}{=} i\hbar$$

$$(x\hat{p} - \hat{p}x)\, f(x) = -i\hbar\left(x\frac{d}{dx} - \frac{d}{dx}x\right)f$$

$$= -i\hbar\left(\cancel{x\frac{df}{dx}} - f - \cancel{x\frac{df}{dx}}\right)$$

$$= i\hbar\, f(x)$$

$$\underbrace{\forall f}$$

$$[x, \hat{p}] = i\hbar$$

$$\underbrace{(i\hat{p} + m\omega x)}\cdot\underbrace{(-i\hat{p} + m\omega x)}$$

$$= 2m\hat{H} + m\hbar\omega$$

$$\begin{cases} a_- \equiv \dfrac{1}{\sqrt{2m\hbar\omega}}\left(i\hat{p} + m\omega x\right) \\[2ex] a_+ \equiv \dfrac{1}{\sqrt{2m\hbar\omega}}\left(-i\hat{p} + m\omega x\right) \end{cases}$$

$$\underbrace{2m}\hbar\omega \; a_- \, a_+ = \underbrace{2m}\,\hat{H} + m\,\underline{\underline{\hbar\omega}}$$

$$\div \; 2m$$

$$(1) \quad \boxed{\hat{H} = \hbar\omega\left(a_- a_+ - \frac{1}{2}\right)}$$

$$2m\hbar\omega \, a_+ \, a_- = 2m\,\hat{H} - m\,\hbar\omega$$

$$\div \; 2m$$

$$(2) \quad \boxed{\hat{H} = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)}$$

$$a_- a_+ - \frac{1}{2} = a_+ a_- + \frac{1}{2}$$

$\downarrow$

$$a_- a_+ - a_+ a_- = 1$$

$$\Rightarrow \quad [a_-, a_+] = 1$$

$$\hookrightarrow \quad \hat{H}\psi = E\psi$$

$$\hbar\omega\left(a_+ a_- + \frac{1}{2}\right)\psi = E\psi$$

EQUIV

$$\hbar\omega\left(a_- a_+ - \frac{1}{2}\right)\psi = E\psi$$

$\rightsquigarrow \psi \curvearrowright$ ENERGIE E

$a_+\psi$: LÖSUNG MIT ENERGIE $E+\hbar\omega$

$a_-\psi$: LÖSUNG MIT ENERGIE $E-\hbar\omega$

$$(a_+ \psi) \;\; ?$$

$$\hat{H}(a_+\psi) \qquad \hat{H} = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)$$

$$= \hbar\omega\left(a_+a_- + \frac{1}{2}\right)a_+\psi$$

$$= \hbar\omega\left(a_+a_-a_+ + \frac{a_+}{2}\right)\psi$$

$$= \hbar\omega\, a_+\left(\underbrace{a_-a_+}_{a_+a_- + 1} + \frac{1}{2}\right)\psi$$

$$= a_+\left(\hat{H} + \hbar\omega\right)\psi$$

$$= a_+\left(E + \hbar\omega\right)\psi$$

$$= (E + \hbar\omega)(a_+\psi)$$

$(a_-\psi)$?

$\hat{H}(a_-\psi)$

$\hat{H} = \hbar\omega\left(a_- a_+ - \frac{1}{2}\right)$

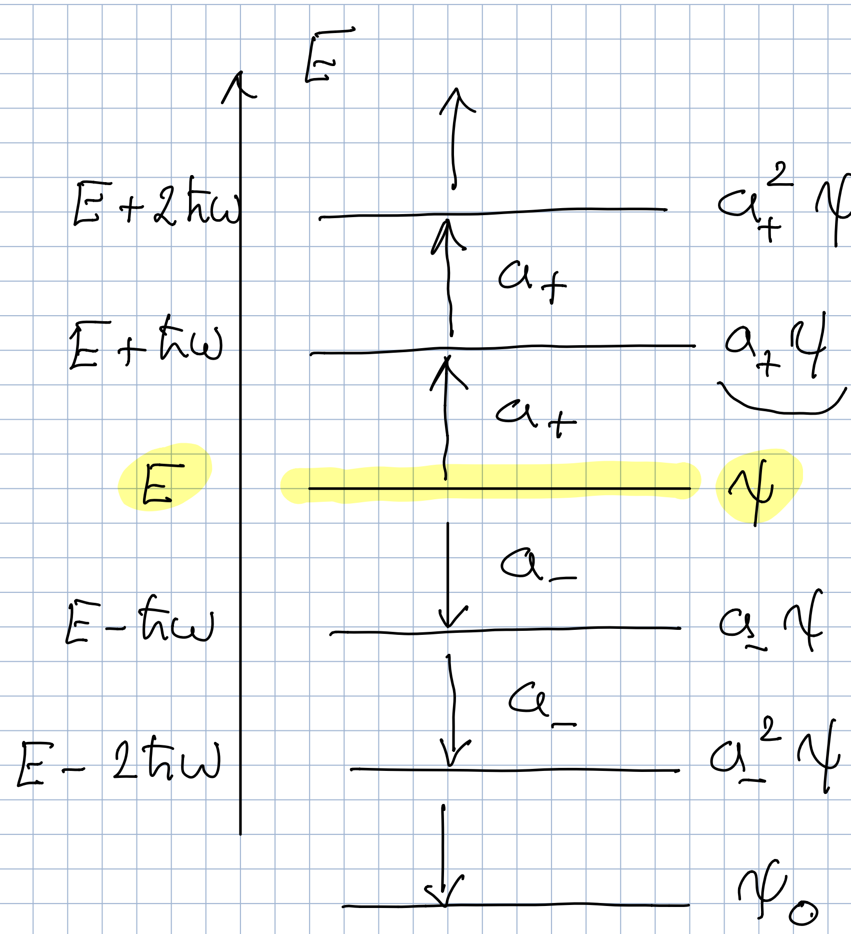
$$= \hbar\omega\left(a_- a_+ a_- - \frac{a_-}{2}\right)\psi$$

$$= a_- \underbrace{\hbar\omega\left(a_+ a_- - \frac{1}{2}\right)}_{\hat{H} - \hbar\omega}\psi$$

$$= a_-\left(\hat{H} - \hbar\omega\right)\psi$$

$$= a_-\left(E - \hbar\omega\right)\psi$$

$$= \left(E - \hbar\omega\right)\left(a_-\psi\right)$$



a_+ : 'RAISING' OPERATOR

a_- : 'LOWERING' OPERATOR.

ψ_0 : GRUNDZUSTAND

$$\rightsquigarrow \quad \underline{\underline{a_- \psi_0 = 0}} \qquad \hat{H}\,(a_-\psi_0) = 0$$

$$\frac{1}{\sqrt{2m\hbar\omega}} \left(i\hat{p} + m\omega x \right) \psi_0 = 0$$

$$\left(\hbar \frac{d}{dx} + m\omega x \right) \psi_0 = 0$$

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar} x \,\psi_0$$

$$\psi_0 = C \, e^{-\left(\frac{m\omega}{2\hbar}\right) x^2}$$

$$1 = \int_{-\infty}^{+\infty} dx \, |\psi_0(x)|^2$$

$$= |C|^2 \underbrace{\int_{-\infty}^{+\infty} dx \;\; e^{-\overbrace{\left(\frac{m\omega}{\hbar}\right)}^{\lambda} x^2}}_{\sqrt{\frac{\pi}{\lambda}}}$$

$$C = \left(\frac{\lambda}{\pi}\right)^{1/4} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$$

ENERGIE

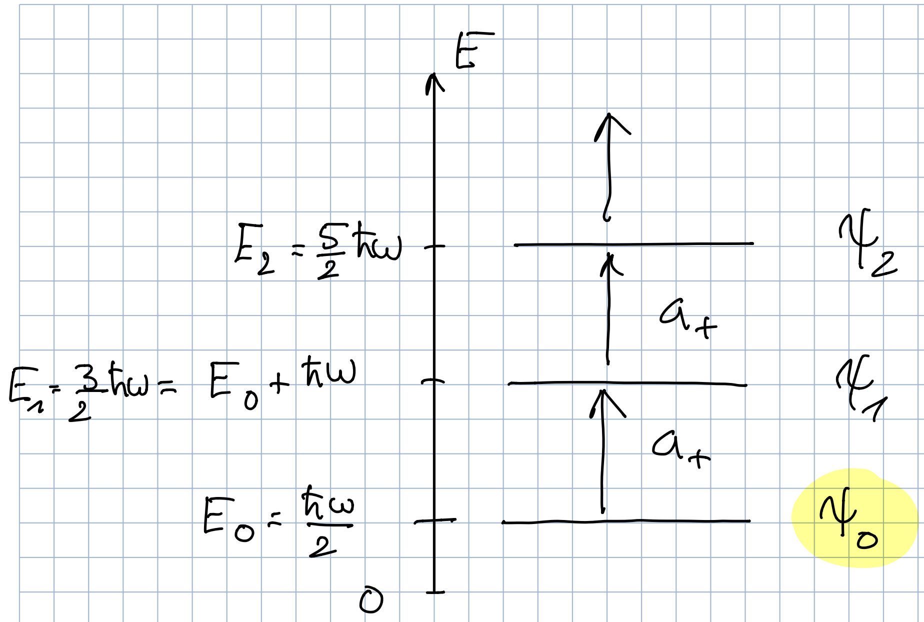
$$E_0 = \langle \hat{H} \rangle = \int dx \; \psi_0^*(x)\, \hat{H}\, \psi_0(x)$$

$$\hat{H} = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)$$

$$E_0 \psi_0 = \hat{H}\psi_0 = \hbar\omega\left(\cancel{a_+ a_-} + \frac{1}{2}\right)\psi_0$$

$$= \frac{\hbar\omega}{2}\,\psi_0$$

$$E_0 = \frac{\hbar\omega}{2} \quad \left(\xrightarrow[\hbar \to 0]{} 0 \right)$$



ψ_1 : 1° ANGEREGTE ZUSTAND

$$\psi_1(x) = A_1 \left(a_+ \psi_0\right)$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$$

$$\psi_1 = A_1 \frac{1}{\sqrt{2m\hbar\omega}} \left(-i\hat{p} + m\omega x\right) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$$

$$\left(-\cancel{\hbar}\frac{d}{dx} + m\omega x\right)$$

$$\downarrow$$

$$\underbrace{-\frac{m\omega}{\cancel{\hbar}}x}_{2m\omega x}$$

$$\psi_1 = A_1 \frac{1}{\sqrt{2m\hbar\omega}} \left(2m\omega x\right) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$$

$$\uparrow$$

$$1 = \int dx \, |\psi_1|^2$$

$$= A_1^2 \left(\frac{\cancel{2}m\omega}{\cancel{\hbar}}\right) \cancel{\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}} \underbrace{\int dx \, x^2 \, e^{-\overset{\lambda}{\left(\frac{m\omega}{\hbar}\right)}x^2}}_{\underbrace{\frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\,\frac{1}{\lambda}}_{\frac{1}{\cancel{2}}\cancel{\left(\frac{\pi\hbar}{m\omega}\right)^{1/2}}\,\cancel{\frac{\hbar}{m\omega}}}}$$

$$\downarrow$$

$$\underline{\underline{A_1 = 1}}$$

$$\Rightarrow \quad \psi_n(x) = A_n \left(a_+^n \psi_0 \right)$$

$$\hat{H} \psi_n = E_n \psi_n$$

$$= \hbar\omega \left(n + \frac{1}{2} \right) \psi_n$$

$$\rightarrow \quad \boxed{A_n = \frac{1}{\sqrt{n!}}}$$

$$\hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$

$$\hat{H} \psi_n = \hbar\omega \left(n + \frac{1}{2} \right) \psi_n$$

$$\Downarrow$$

$$a_+ a_- \psi_n = n \psi_n$$

ZÄHL OPERATOR

$$n = 0, 1, 2, 3 \ldots$$

$$\longrightarrow \quad a_- a_+ \psi_n = \left(a_+ a_- + 1 \right) \psi_n = (n+1) \psi_n$$

$$[a_-, a_+] = 1$$

$$\rightsquigarrow \quad \underline{\underline{a_+ \psi_n}} = c_n \, \underline{\underline{\psi_{n+1}}}$$

$$\int_{-\infty}^{+\infty} dx \, \underbrace{(a_+ \psi_n)^*}\, \underline{\underline{(a_+ \psi_n)}} = |c_n|^2 \underbrace{\int dx \, |\psi_{n+1}|^2}_{\underline{1}}$$

$$a_+ \psi_n^* = \frac{1}{\sqrt{2m\hbar\omega}} \left(-\hbar \frac{d}{dx} + m\omega x \right) \psi_n^*$$

P.I

$$\int_{-\infty}^{+\infty} dx \, \psi_n^* \, \frac{1}{\sqrt{2m\hbar\omega}} \left(+\hbar \frac{d}{dx} + m\omega x \right) a_+ \psi_n$$

$$= \int_{-\infty}^{+\infty} dx \, \psi_n^* \, \underbrace{a_- a_+ \psi_n}_{(n+1)\,\psi_n}$$

$$= (n+1)$$

$$= |c_n|^2 \quad \Longrightarrow \quad c_n = \sqrt{n+1}$$

$$\begin{cases} a_+ \psi_n = \sqrt{n+1}\ \psi_{n+1} \\ a_- \psi_n = \sqrt{n}\ \psi_{n-1} \end{cases}$$

$$\psi_0$$

$$\psi_1 = a_+ \psi_0$$

$$\psi_2 = \frac{1}{\sqrt{2}} a_+ \psi_1 = \frac{1}{\sqrt{2}} a_+^2 \psi_0$$

$$\psi_3 = \frac{1}{\sqrt{3}} a_+ \psi_2 = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} a_+^3 \psi_0 = \frac{1}{\sqrt{3!}} a_+^3 \psi_0$$

$$\vdots$$

$$\psi_n = \frac{1}{\sqrt{n!}} a_+^n \psi_0$$

$\hookrightarrow$ LÖSUNGEN SIND ORTHOGONAL:

$$\int dx \; \psi_m^*(x)\, \psi_n(x) = \delta_{nm}$$

$$\rightsquigarrow \int_{-\infty}^{+\infty} dx \; \psi_m^* \underbrace{(a_+ a_-)}\, \psi_n$$

$$= n \int dx \; \psi_m^* \psi_n$$

$$\int_{-\infty}^{+\infty} dx \; (a_- \psi_m)^* (a_- \psi_n)$$

$$\int_{-\infty}^{+\infty} dx \; (\underbrace{a_+ a_- \psi_m}_{m\,\psi_m})^* \psi_n$$

$$= m \int dx \; \psi_m^* \psi_n$$

$$n \neq m \implies \int dx \; \psi_m^* \psi_n = 0$$

ORTHOGONAL

$$\begin{cases} \Psi_0(x,t) = \psi_0(x)\, e^{-\frac{i}{\hbar}E_0 t} \\ \Psi_n(x,t) = \psi_n(x)\, e^{-\frac{i}{\hbar}E_n t} \end{cases}$$