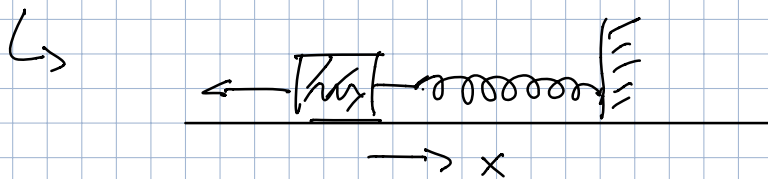


⇒ VORLESUNG 4 QM

↳ 2.3 HARMONISCHE OSCILLATOR

• KLASSISCHE H.O.



$$F = -kx$$

$$m \frac{d^2x}{dt^2} = -kx$$

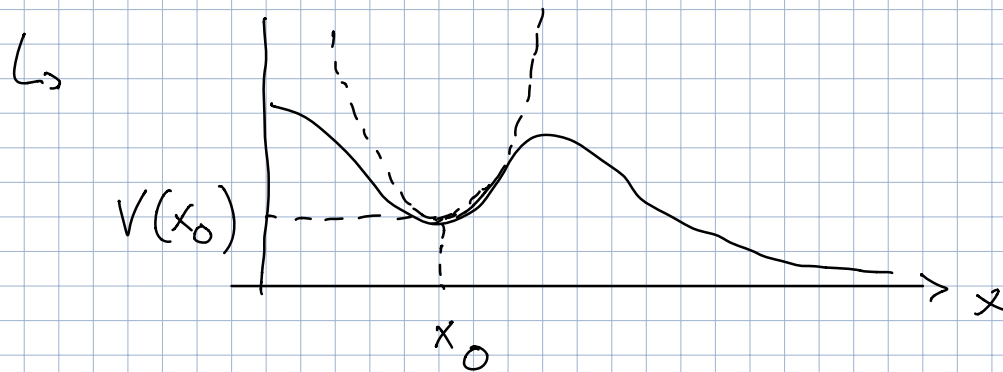
$$x(t) = A \sin(\omega t) + B \cos(\omega t)$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$F = - \frac{dV}{dx}$$

$$V(x) = \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2$$

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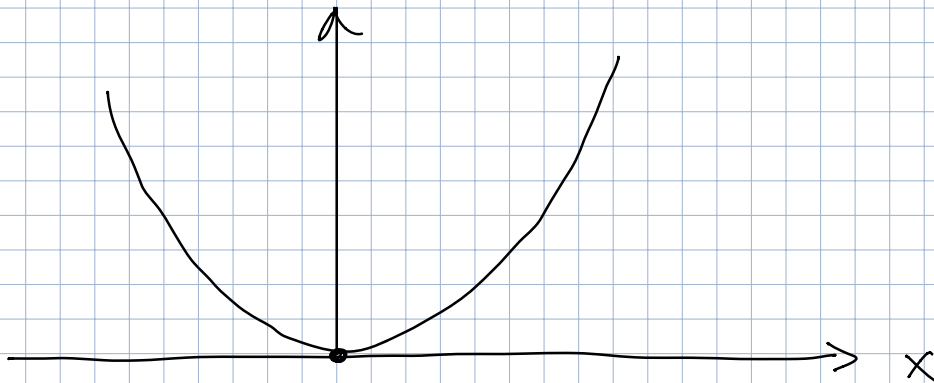
$$V(x) = V(x_0) + \cancel{V'(x_0)(x-x_0)} + \frac{1}{2} V''(x_0) (x-x_0)^2 + \dots$$

$$V(x) - V(x_0) = \frac{1}{2} \underbrace{V''(x_0)}_k (x-x_0)^2 + \dots$$

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• QUANTUM H.O.

$$V(x) = \frac{1}{2} m \omega^2 x^2$$



$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi = E \psi$$

REIHEN ANSATZ



ALGEBRAISCHE METHODE

$$\hat{p} = -i\hbar \frac{d}{dx}$$

$$\frac{\hat{p}^2}{2m} \psi + \frac{1}{2} m \omega^2 x^2 \psi = E \psi$$

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$$\Rightarrow \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$\hat{H} \psi = E \psi$$

$$(i\hat{p} + m\omega x) \cdot (-i\hat{p} + m\omega x)$$

$$= \hat{p}^2 + i m \omega (\hat{p} x - x \hat{p}) + m^2 \omega^2 x^2$$

$$= \underbrace{2m \hat{H}} + \underbrace{i m \omega (\hat{p} x - x \hat{p})}_{-i\hbar}$$

$$\left. \begin{aligned} x^2 - a^2 &= (x - a)(x + a) \end{aligned} \right\}$$

$$x^2 + b^2 = x^2 - (ib)^2$$

$$= (x - ib)(x + ib)$$

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$$x \hat{p} - \hat{p} x = [x, \hat{p}] \stackrel{?}{=} i\hbar$$

$$\begin{aligned} (x \hat{p} - \hat{p} x) f(x) &= -i\hbar \left(x \frac{d}{dx} - \frac{d}{dx} x \right) f \\ &= -i\hbar \left(\cancel{x \frac{df}{dx}} - f - \cancel{x \frac{df}{dx}} \right) \\ &= i\hbar f(x) \end{aligned}$$

$\forall f$

$$[x, \hat{p}] = i\hbar$$

$$\begin{aligned} & \underbrace{(i\hat{p} + m\omega x)}_{\hat{H}} \cdot \underbrace{(-i\hat{p} + m\omega x)}_{\hat{H}} \\ &= \underline{\underline{2m\hat{H}}} + \underline{\underline{m\hbar\omega}} \end{aligned}$$

$$\begin{cases} a_- \equiv \frac{1}{\sqrt{2m\hbar\omega}} (i\hat{p} + m\omega x) \\ a_+ \equiv \frac{1}{\sqrt{2m\hbar\omega}} (-i\hat{p} + m\omega x) \end{cases}$$

$$\underline{2m\hbar\omega} a_- a_+ = \underline{2m\hat{H}} + \underline{m\hbar\omega}$$

$$\div 2m$$

$$(1) \hat{H} = \hbar\omega \left(a_- a_+ - \frac{1}{2} \right)$$

$$2m\hbar\omega a_+ a_- = 2m\hat{H} - m\hbar\omega$$

$$\div 2m$$

$$(2) \hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$

$$a_- a_+ - \frac{1}{2} = a_+ a_- + \frac{1}{2}$$

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$$a_- a_+ - a_+ a_- = 1$$

$$\Rightarrow [a_-, a_+] = 1$$

$$\hookrightarrow \hat{H} \psi = E \psi$$

$$\hbar \omega \left(a_+ a_- + \frac{1}{2} \right) \psi = E \psi$$

EQUIV

$$\hbar \omega \left(a_- a_+ - \frac{1}{2} \right) \psi = E \psi$$

ψ \rightarrow ENERGIE E

$a_+ \psi$: LÖSUNG MIT ENERGIE $E + \hbar \omega$

$a_- \psi$: LÖSUNG MIT ENERGIE $E - \hbar \omega$

$$(a_+ \psi)_0 ?$$

$$\hat{H}(a_+ \psi) \quad \hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$
$$= \hbar\omega \left(a_+ a_- + \frac{1}{2} \right) a_+ \psi$$

$$= \hbar\omega \left(a_+ a_- a_+ + \frac{a_+}{2} \right) \psi$$

$$= \hbar\omega a_+ \left(a_- a_+ + \frac{1}{2} \right) \psi$$

$$\underline{\underline{a_+ a_- + 1}}$$

$$= a_+ \left(\hat{H} + \hbar\omega \right) \psi$$

$$= a_+ (E + \hbar\omega) \psi$$

$$= (E + \hbar\omega) (a_+ \psi)$$

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$(a_- \psi) ?$

$$\hat{H} (a_- \psi) \quad \left. \begin{array}{l} \nearrow \\ \searrow \end{array} \right\} \hat{H} = \hbar\omega \left(a_- a_+ - \frac{1}{2} \right)$$

$$= \hbar\omega \left(a_- a_+ a_- - \frac{a_-}{2} \right) \psi$$

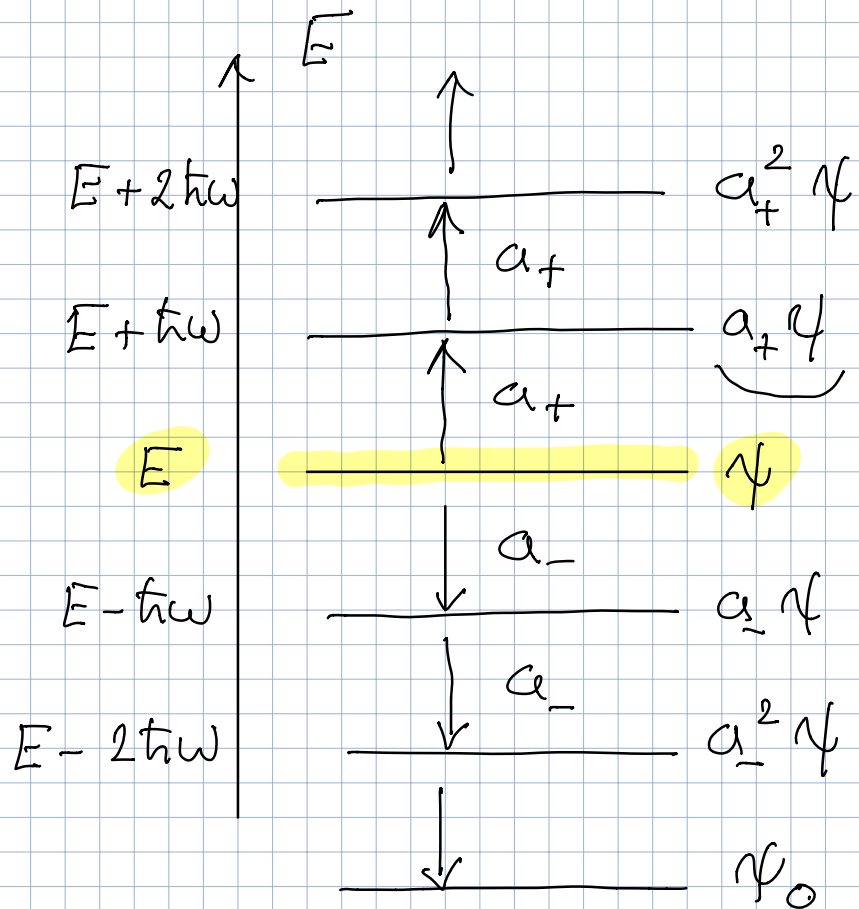
$$= a_- \underbrace{\hbar\omega \left(a_+ a_- - \frac{1}{2} \right)}_{\hat{H} - \hbar\omega} \psi$$

$$= a_- \left(\hat{H} - \hbar\omega \right) \psi$$

$$= a_- \left(E - \hbar\omega \right) \psi$$

$$= (E - \hbar\omega) (a_- \psi)$$

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a_+ : 'RAISING' OPERATOR
 a_- : 'LOWERING' OPERATOR.

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ψ_0 : GRUNDZUSTAND

$$\Rightarrow \underline{\underline{a_- \psi_0 = 0}} \quad \hat{H}(a_- \psi_0) = 0$$

$$\frac{1}{\sqrt{2m\hbar\omega}} \left(i\hat{p} + m\omega x \right) \psi_0 = 0$$

$$\left(\hbar \frac{d}{dx} + m\omega x \right) \psi_0 = 0$$

$$\frac{d\psi_0}{dx} = - \frac{m\omega}{\hbar} x \psi_0$$

$$\psi_0 = C e^{-\left(\frac{m\omega}{2\hbar}\right) x^2}$$

$$1 = \int_{-\infty}^{+\infty} dx |\psi_0(x)|^2$$
$$= |C|^2 \int_{-\infty}^{+\infty} dx e^{-\frac{\lambda}{\hbar} x^2}$$

$$C = \left(\frac{\lambda}{\pi} \right)^{1/4} = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4}$$

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$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar} \right) x^2}$$

ENERGIE

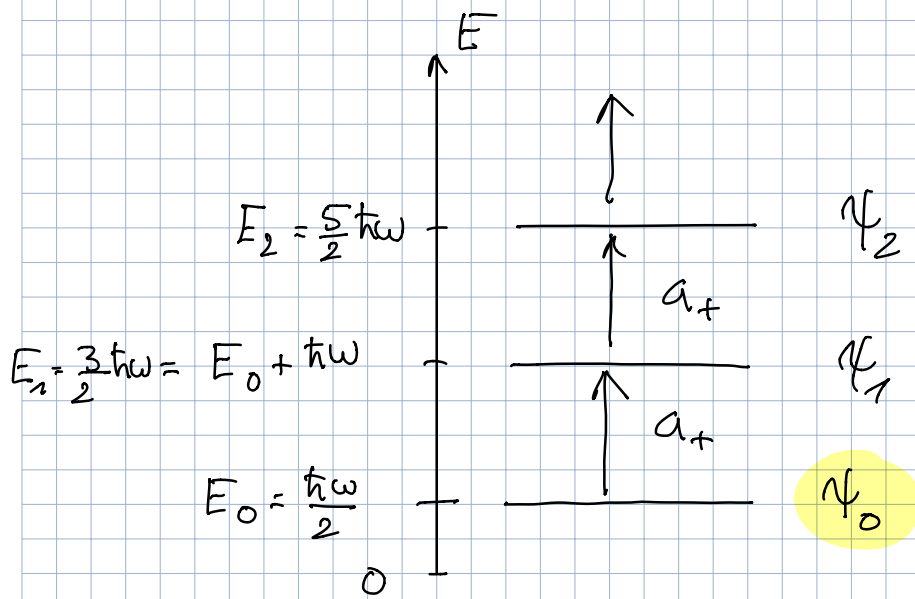
$$E_0 = \langle \hat{H} \rangle = \int dx \psi_0^*(x) \hat{H} \psi_0(x)$$

$$\hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$

$$\begin{aligned} E_0 \psi_0 &= \hat{H} \psi_0 = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right) \psi_0 \\ &= \frac{\hbar\omega}{2} \psi_0 \end{aligned}$$

$$E_0 = \frac{\hbar\omega}{2} \quad \left(\xrightarrow{\hbar \rightarrow 0} 0 \right)$$

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ψ_1 : 1^o ANGEREGTE ZUSTAND

$$\psi_1(x) = A_1 (a_+ \psi_0)$$

$$\psi_0(x) = \left(\frac{m\omega}{\hbar} \right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar} \right) x^2}$$

$$\psi_1 = A_1 \frac{1}{\sqrt{2m\hbar\omega}} \left(-i\hat{p} + m\omega x \right) \left(\frac{m\omega}{\hbar} \right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar} \right) x^2}$$

$$\left(-\cancel{\hbar} \frac{d}{dx} + m\omega x \right)$$

↓

$$-\frac{m\omega x}{\cancel{\hbar}}$$

$$2m\omega x$$

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$$\psi_1 = A_1 \frac{1}{\sqrt{2m\hbar\omega}} (2m\omega x) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$$

$$1 = \int dx |\psi_1|^2$$

$$= A_1^2 \left(\frac{2m\omega}{\hbar}\right) \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int dx x^2 e^{-\left(\frac{m\omega}{\hbar}\right)x^2}$$

$$\frac{1}{2} \sqrt{\frac{\hbar}{m\omega}} \frac{1}{\hbar}$$

$$\frac{1}{2} \left(\frac{\pi\hbar}{m\omega}\right)^{1/2} \frac{\hbar}{m\omega}$$

↓

$$\underline{\underline{A_1 = 1}}$$

$$\Rightarrow \psi_m(x) = A_m (a_+^m \psi_0)$$

$$\hat{H} \psi_m = E_m \psi_m$$

$$= \hbar\omega \left(m + \frac{1}{2} \right) \psi_m$$

$$\Rightarrow \parallel A_m = \frac{1}{\sqrt{m!}}$$

$$\hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$

$$\hat{H} \psi_m = \hbar\omega \left(m + \frac{1}{2} \right) \psi_m$$

$$\Downarrow$$
$$a_+ a_- \psi_m = m \psi_m$$

ZÄHL OPERATOR

$$m = 0, 1, 2, 3, \dots$$

$$\rightarrow a_- a_+ \psi_m = (a_+ a_- + 1) \psi_m = (m+1) \psi_m$$

$$[a_-, a_+] = 1$$

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$$\leadsto \underline{a_+ \psi_m} = c_n \underline{\psi_{m+1}}$$

$$\int_{-\infty}^{+\infty} dx \underbrace{(a_+ \psi_m)^*}_{\uparrow} \underbrace{(a_+ \psi_m)}_{\underline{\quad}} = |c_n|^2 \underbrace{\int dx |\psi_{m+1}|^2}_{\underline{1}}$$

$$a_+ \psi_m^* = \frac{1}{\sqrt{2m\hbar\omega}} \left(-\hbar \frac{d}{dx} + m\omega x \right) \psi_m^*$$

P.I

$$\int_{-\infty}^{+\infty} dx \psi_m^* \frac{1}{\sqrt{2m\hbar\omega}} \left(+\hbar \frac{d}{dx} + m\omega x \right) a_+ \psi_m$$

$$= \int_{-\infty}^{+\infty} dx \psi_m^* \underbrace{a_- a_+ \psi_m}_{(n+1) \psi_m}$$

$$= (n+1)$$

$$= |c_n|^2 \quad \Rightarrow \quad c_n = \sqrt{n+1}$$

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$$\begin{cases} a_+ \psi_m = \sqrt{m+1} \psi_{m+1} \\ a_- \psi_m = \sqrt{m} \psi_{m-1} \end{cases}$$

$$\psi_0$$

$$\psi_1 = a_+ \psi_0$$

$$\psi_2 = \frac{1}{\sqrt{2}} a_+ \psi_1 = \frac{1}{\sqrt{2}} a_+^2 \psi_0$$

$$\psi_3 = \frac{1}{\sqrt{3}} a_+ \psi_2 = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} a_+^3 \psi_0 = \frac{1}{\sqrt{3!}} a_+^3 \psi_0$$

⋮

$$\psi_m = \frac{1}{\sqrt{m!}} a_+^m \psi_0$$

↳ LÖSUNGEN SIND ORTHOGONAL:

$$\int dx \psi_m^*(x) \psi_m(x) = \delta_{mm}$$

$$\rightsquigarrow \int_{-\infty}^{+\infty} dx \psi_m^* (a_+ a_-) \psi_m$$

$$= m \int_{-\infty}^{+\infty} dx \psi_m^* \psi_m$$

$$\int_{-\infty}^{+\infty} dx (a_- \psi_m)^* (a_- \psi_m)$$

$$\int_{-\infty}^{+\infty} dx \underbrace{(a_+ a_- \psi_m)^*}_{m \psi_m^*} \psi_m$$

$$= m \int_{-\infty}^{+\infty} dx \psi_m^* \psi_m$$

$$m \neq m \Rightarrow \int dx \psi_m^* \psi_m = 0$$

ORTHOGONAL

$$\left\{ \begin{array}{l} \Psi_0(x, t) = \psi_0(x) e^{-\frac{i}{\hbar} E_0 t} \\ \Psi_n(x, t) = \psi_n(x) e^{-\frac{i}{\hbar} E_n t} \end{array} \right.$$