⟹ VORLESUNG 3 QM

KAPITEL 2 : ZEIT-UNABHÄNGIGE SCHRÖDINGER GL.

↳ 2.1 STATIONÄRE ZUSTÄNDE

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$

$$\Psi(x,t)$$

$$\underline{V(x,t) = V(x)}$$

TRENNUNG DER VARIABLEN

$$\Psi(x,t) = \psi(x)\,\varphi(t)$$

$$-\frac{\hbar^2}{2m}\varphi\frac{d^2\psi}{dx^2} + V\varphi\psi = i\hbar\psi\frac{d\varphi}{dt}$$

$$\div\ \varphi\psi$$

$$\underbrace{-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V(x)}_{\text{FUNKTION VON } x} = \underbrace{i\hbar\frac{1}{\varphi}\frac{d\varphi}{dt}}_{\text{FUNKTION VON } t}$$

$$= E \quad (\text{KONSTANTE})$$

$$\begin{cases} i\hbar \dfrac{1}{\varphi}\dfrac{d\varphi}{dt} = E & (1) \\ -\dfrac{\hbar^2}{2m}\dfrac{1}{\psi}\dfrac{d^2\psi}{dx^2} + V(x) = E & (2) \end{cases}$$

$$(1) \qquad \frac{d\varphi}{dt} = -\frac{i}{\hbar}E\varphi$$

$$\varphi(t) = \underbrace{C}_{\sim} e^{-\frac{i}{\hbar}Et}$$

$$C = 1$$

$$\Psi(x,t) = \varphi(t)\,\psi(x)$$

$$\int dx\, |\Psi(x,t)|^2 = 1$$

$$(2) \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

ZEIT-UNABHÄNGIGE SCHR. GL.

$$\downarrow$$

$$\psi(x)$$

$$\Psi(x,t) = \psi(x)\, e^{-\frac{i}{\hbar}Et}$$

$$|\Psi(x,t)|^2 = |\psi(x)|^2$$

ZEITUNABHÄNGIG

STATIONÄRE ZUSTAND

$$\langle x \rangle = \int dx\, \underbrace{|\Psi(x,t)|^2}\, x$$

$$= \int dx\, |\psi(x)|^2\, x$$

ZEITUNABH.

$$m\frac{d}{dt}\langle x \rangle = \langle p \rangle = 0$$

↳ STATIONÄRER ZUSTAND
ENTSPRICHT FESTER ENERGIE

- KL $$H(x,p) = \frac{p^2}{2m} + V(x)$$

- QM $$p \longrightarrow -i\hbar \frac{\partial}{\partial x}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\hat{H}\psi = E\psi$$

$$\langle \hat{H} \rangle = \int dx\, \psi^* \underbrace{\hat{H}\psi}_{E\psi}$$

$$= E \int dx\, |\psi(x)|^2$$

$$= E$$

↑

ENERGIE DES SYSTEMS

↳ ALLGEMEINE LÖSUNG

$$\hat{H}\, \psi_{\underbrace{n}} = E_n\, \psi_n$$

↳ DISKRET, CONT.

$$\Psi(x,t) = \sum_n c_n\, \psi_n(x)\, e^{-\frac{i}{\hbar} E_n t}$$

ANFANGSBEDINGUNG

$$\Psi(x,0) = \sum_n c_n\, \psi_n(x)$$

$$\int dx\, \psi_m^*(x)\, \Psi(x,0) = \sum_n c_n \underbrace{\int dx\, \psi_m^*(x)\, \psi_n(x)}_{\delta_{nm}}$$

$$= c_m$$

BEISPIEL $\quad \psi_1, \psi_2$

$$\begin{array}{cc} \psi_1 & \psi_2 \\ \downarrow & \downarrow \\ E_1 & E_2 \end{array}$$

$$\Psi(x,t) = c_1 \psi_1(x) e^{-\frac{i}{\hbar}E_1 t} + c_2 \psi_2(x) e^{-\frac{i}{\hbar}E_2 t}$$

$$\underline{\underline{\Psi(x,0)}} = c_1 \psi_1(x) + c_2 \psi_2(x)$$

FÜR c_1, c_2, ψ_1, ψ_2 REAL

$$|\Psi(x,t)|^2 = c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + 2 c_1 c_2 \psi_1 \psi_2 \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)$$

$$\frac{E_1 - E_2}{\hbar} \equiv \omega_{12}$$

$\underline{c_2 = 0}$ $\qquad |\Psi(x,t)|^2 = c_1^2 \psi_1^2(x)$

MAX $\qquad |\Psi(x,t)|^2_{MAX} = (c_1 \psi_1 + c_2 \psi_2)^2$

MIN $\qquad |\Psi(x,t)|^2_{MIN} = (c_1 \psi_1 - c_2 \psi_2)^2$

$$\underline{\Psi(x,t)} = c_1 \psi_1 e^{-\frac{i}{\hbar}E_1 t} + c_2 \psi_2 e^{-\frac{i}{\hbar}E_2 t}$$

$$|\Psi(x,t)|^2 = \left(c_1 \psi_1 e^{-\frac{i}{\hbar}E_1 t} + c_2 \psi_2 e^{-\frac{i}{\hbar}E_2 t} \right)$$

$$\cdot \left(c_1^* \psi_1^* e^{\frac{i}{\hbar}E_1 t} + c_2^* \psi_2^* e^{\frac{i}{\hbar}E_2 t} \right)$$

$$= |c_1|^2 |\psi_1|^2 + |c_2|^2 |\psi_2|^2$$

$$+ c_1 c_2^* \psi_1 \psi_2^* e^{-\frac{i}{\hbar}(E_1 - E_2)t}$$

$$+ c_1^* c_2 \psi_1^* \psi_2 e^{\frac{i}{\hbar}(E_1 - E_2)t}$$

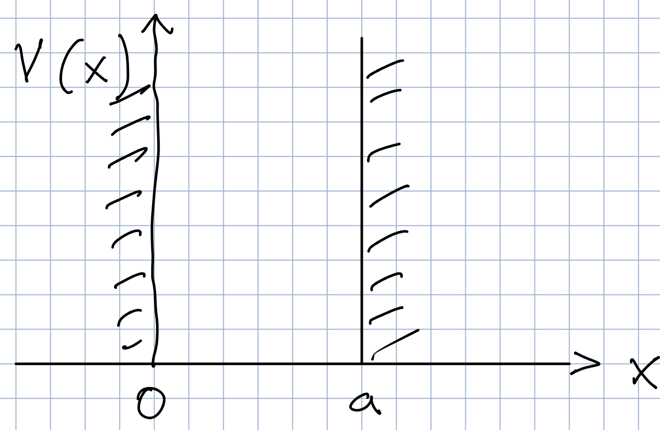
$$= |c_1|^2 |\psi_1|^2 + |c_2|^2 |\psi_2|^2$$

$$+ 2\,\mathrm{Re}\left(c_1 c_2^* \psi_1 \psi_2^* e^{-\frac{i}{\hbar}(E_1 - E_2)t} \right)$$

$\downarrow$ c_1, c_2, ψ_1, ψ REAL

$$2 c_1 c_2 \psi_1 \psi_2 \cos\left(\frac{E_1 - E_2}{\hbar} t \right)$$

↳ 2.2 UNENDLICHER POTENTIALTOPF



$$V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & \end{cases}$$

$x < 0 \;,\; x > a \qquad \psi(x) = 0$

$0 \leq x \leq a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$\longrightarrow \quad \frac{d^2\psi}{dx^2} = -k^2\psi$$

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

$$\psi(0) = 0 \quad , \quad \psi(a) = 0$$

$$\downarrow$$

$$\underline{B = 0}$$

$$\rightarrow \quad \psi(x) = A \sin kx$$

$$0 = \psi(a) = A \sin(\underline{\underline{ka}})$$

$$ka = 0, \pm\pi, \pm 2\pi$$

1) $\cancel{k=0}$? NICHT NORMIERBAR

2) $k = \pi \longleftrightarrow \cancel{k = -\pi}$

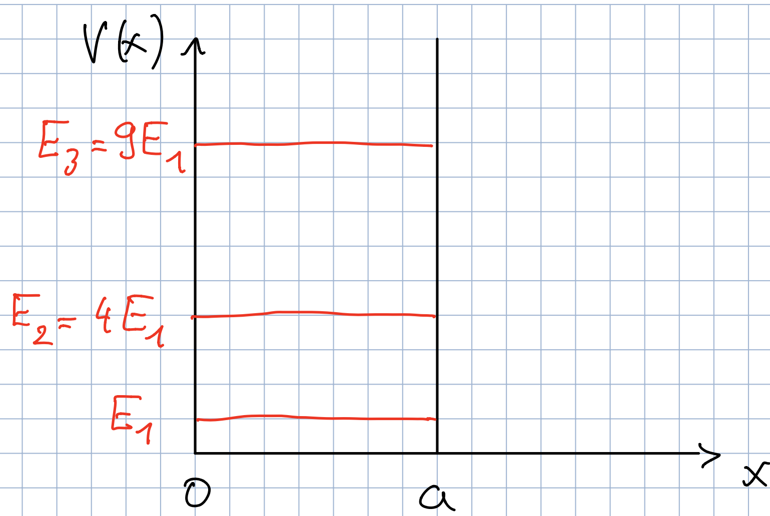
$$k_n = n\frac{\pi}{a} \qquad n = 1, 2, 3, \ldots$$

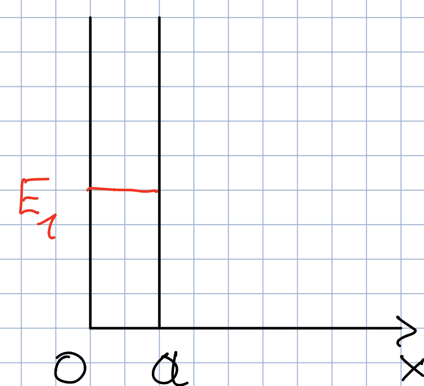
$$\rightarrow \quad \psi_n(x) = A \sin\left(n\frac{\pi}{a}x\right)$$

$$\underline{E_n = \frac{\hbar^2}{2m} k_n^2 = n^2 \frac{\hbar^2\pi^2}{2ma^2}}$$

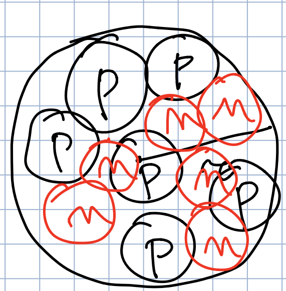
E_1

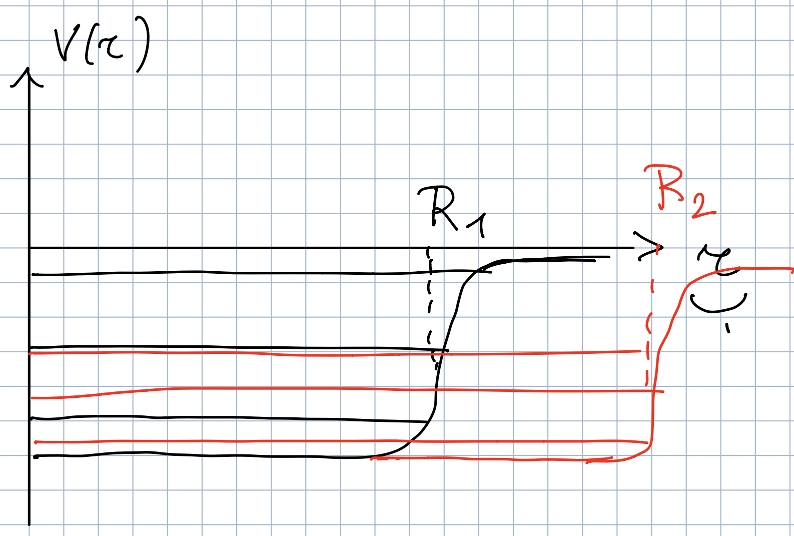
$$n = 1, 2, \ldots$$





^{12}C





^{208}Pb → 82 p
→ 126 n

↳ NORMIERUNG

$$\int_{-\infty}^{+\infty} dx \, |\psi(x)|^2 = 1$$

$$\int_0^a dx \, \underbrace{|\psi(x)|^2}_{|A|^2 \sin^2\left(\frac{n\pi}{a} x\right)} = 1$$

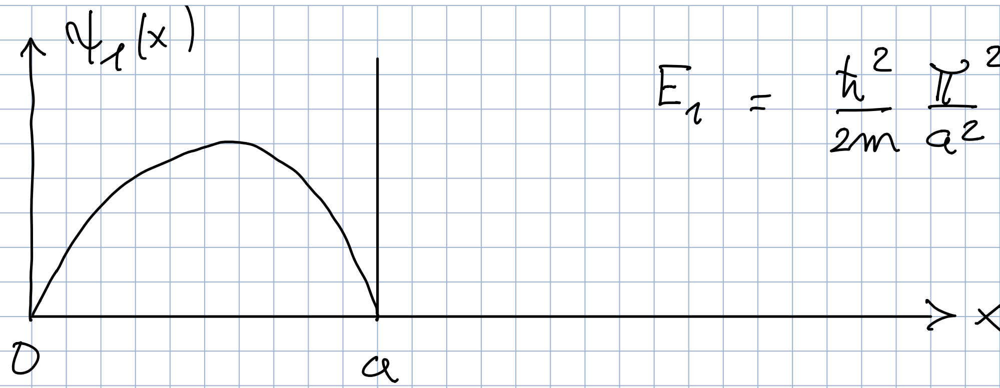
$$\downarrow \qquad \sin^2 y = \frac{1}{2}\left(1 - \cos 2y\right)$$

$$1 = |A|^2 \int_0^a dx \, \frac{1}{2}\left(1 - \cancel{\cos \frac{n\pi 2}{a} x}\right)$$

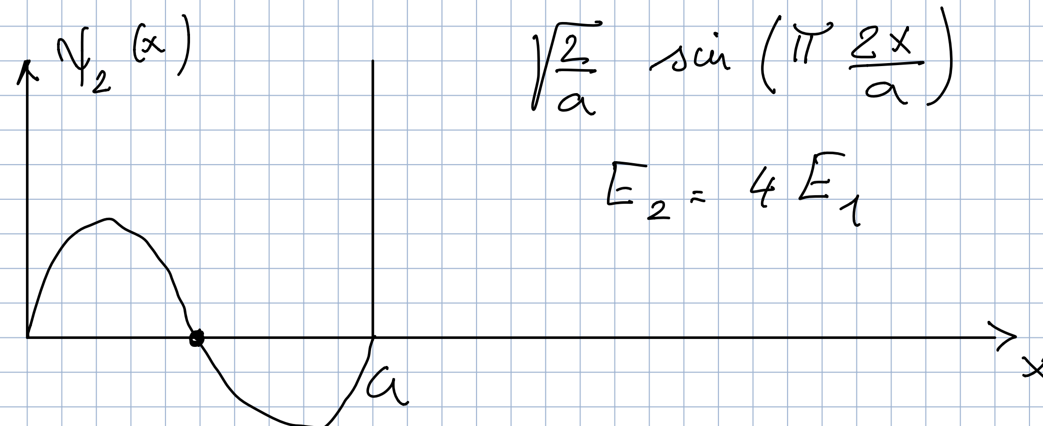
$$= |A|^2 \, \frac{a}{2}$$

$$\hookrightarrow \quad A = \sqrt{\frac{2}{a}}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \, \sin\left(\frac{n\pi}{a} x\right)$$

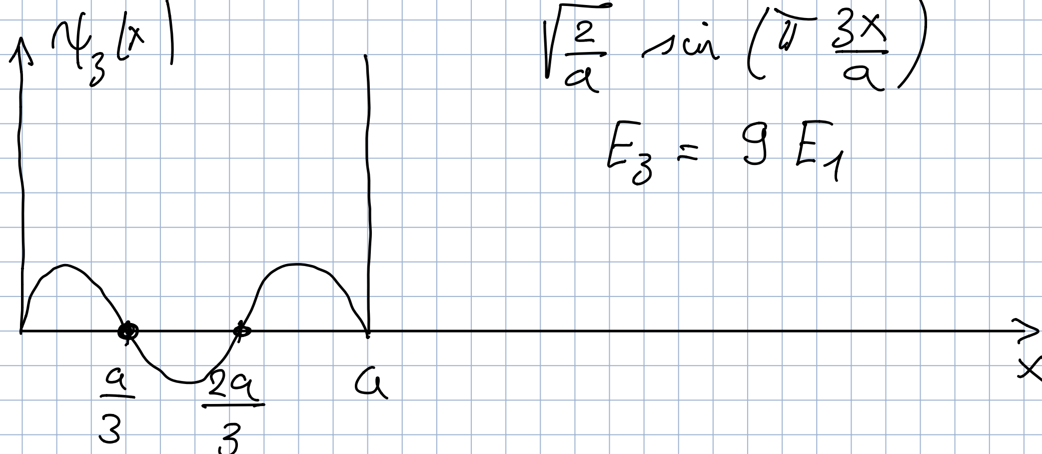


$$E_1 = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}$$



$$\sqrt{\frac{2}{a}}\,\sin\left(\pi\,\frac{2x}{a}\right)$$

$$E_2 = 4\,E_1$$



$$\sqrt{\frac{2}{a}}\,\sin\left(\pi\,\frac{3x}{a}\right)$$

$$E_3 = 9\,E_1$$

ψ_n $(n-1)$ KNOTENPUNKTE

$\psi_2, \psi_3, \ldots$ ANGEREGTE ZUSTÄNDE

ψ_1 GRUNDZUSTAND

→ ORTHONORMALITÄT

$$\int dx \; \psi_m^*(x)\, \psi_n(x) = \delta_{nm}$$

→ VOLLSTÄNDIGKEIT

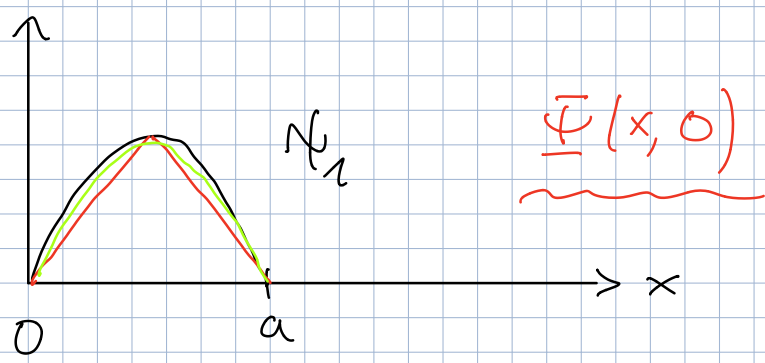
$$\psi(x) = \sum_n c_n \, \psi_n(x)$$

$$= \sum_n c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

$$c_n = \int dx \; \underbrace{\psi(x)}_{t=0} \; \psi_n^*(x)$$

$$\Psi(x,t) = \sum_n c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right) e^{-\frac{i}{\hbar} E_n t}$$

$$\Psi(x,0) \longrightarrow c_n$$



c_n ?

$c_2 = c_4 = \dots = 0.$

$c_1 \gg c_3 > c_5 \quad \dots$

e.g. $\Psi(x,0) = A\,x\,(a-x)$

$c_2 = c_4 = \dots = 0$

$|c_1|^2 \approx 0.999$

$\hookrightarrow \langle \hat{H} \rangle = \sum_n |c_n|^2 E_n$

$= \int dx\, \Psi^* \underbrace{\hat{H}\, \Psi}_{\sum_n c_n E_n \psi_n e^{-\frac{i}{\hbar} E_n t}}$

$$= \sum_n \sum_m c_n c_m^* \underbrace{\int dx\, \psi_m^*(x)\, \psi_n(x)}_{\delta_{nm}} E_n\, e^{-\frac{i}{\hbar}(E_n - E_m)t}$$

$$= \sum_n |c_n|^2 E_n$$

$\hookrightarrow$ σ_x

σ_p $\Longrightarrow$ $\underbrace{\sigma_x \cdot \sigma_p} \geq \frac{\hbar}{2}$

e.g. $n = 1$ $\frac{\hbar}{2}(1.14)$