CHAPTER 2 :

TIME - INDEPENDENT SCHRÖDINGER EQUATION

⇒ 2.1 STATIONARY STATES

- SOLVE SCHRÖDINGER EQ.

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$

FOR TIME-INDEPENDENT POTENTIAL $V(x,t) = V(x)$

- METHOD OF SEPARATION OF VARIABLES. TO SOLVE PARTIAL DIFFERENTIAL EQ.

TRY SOLUTION OF FORM

$$\Psi(x,t) = \psi(x) \cdot \varphi(t)$$

$$\hookrightarrow \frac{\partial^2 \Psi}{\partial x^2} = \frac{d^2\psi}{dx^2} \cdot \varphi$$

$$\hookrightarrow \frac{\partial \Psi}{\partial t} = \psi \cdot \frac{d\varphi}{dt}$$

↳ SCHRÖDINGER EQ.

$$i\hbar\, \psi \frac{d\varphi}{dt} = -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} \varphi + V(x)\, \psi\, \varphi$$

↓ DIVIDE BY $\psi . \varphi$

$$\underbrace{i\hbar \frac{1}{\varphi} \frac{d\varphi}{dt}}_{\text{ONLY FUNCTION OF } t} = \underbrace{-\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2\psi}{dx^2} + V(x)}_{\text{ONLY FUNCTION OF } x}$$

ONLY POSSIBLE IF BOTH ARE CONSTANT

DENOTING CONSTANT BY E

$$\begin{cases} i\hbar \dfrac{1}{\varphi} \dfrac{d\varphi}{dt} = E \\[2ex] -\dfrac{\hbar^2}{2m} \dfrac{1}{\psi} \dfrac{d^2\psi}{dx^2} + V(x) = E \end{cases}$$

∴ USING METHOD OF SEPARATION OF VARIABLES. TO TURN PARTIAL DIFFERENTIAL EQUATION INTO 2 ORDINARY DIFFERENTIAL EQUATIONS

→ ONE FOR $\varphi(t)$

→ ONE FOR $\psi(x)$

$$\hookrightarrow \quad \boxed{i\hbar \frac{\partial \varphi}{\partial t} = E\varphi}$$

GENERAL SOLUTION

$$\varphi(t) = \underset{\substack{\uparrow \\ \text{CONSTANT}}}{C} \cdot e^{-\frac{i}{\hbar}Et}$$

BECAUSE WE WILL NORMALIZE $\Psi = \psi(x) \cdot \varphi(t)$, WE CAN SET (WITHOUT LOSS OF GENERALITY) $C = 1$

$$\varphi(t) = e^{-\frac{i}{\hbar}Et}$$

$\hookrightarrow$ $\psi(x)$ SATISFIES

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi}$$

TIME INDEPENDENT SCHRÖDINGER EQUATION

$\hookrightarrow$ GENERAL SOLUTION OF FORM

$$\boxed{\Psi(x,t) = \psi(x)\, e^{-\frac{i}{\hbar}Et}}$$

- SOLUTION $\Psi(x,t) = \psi(x)\, e^{-\frac{i}{\hbar}Et}$

IS A STATIONARY STATE

↳ PROBABILITY DENSITY IS TIME INDEPENDENT

$$|\Psi(x,t)|^2 = |\psi(x)|^2$$

ALSO EXPECTATION VALUES ARE TIME INDEPENDENT

$$\text{e.g.} \quad \langle x \rangle = \int dx\ x\, |\Psi(x,t)|^2$$
$$= \int dx\ x\ |\psi(x)|^2$$

$$m \frac{d}{dt} \langle x \rangle = \langle p \rangle = 0$$

↳ AVERAGE MOMENTUM IN STATIONARY STATE = 0.

↳ STATIONARY STATE IS STATE OF DEFINITE TOTAL ENERGY E

HAMILTONIAN: CLASSICAL MECH $H(x,p) = \frac{p^2}{2m} + V(x)$

↓

QUANTUM MECH $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

$p \rightarrow -i\hbar \frac{\partial}{\partial x}$

TIME INDEPENDENT SCHRÖDINGER EQ.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi(x) = E\,\psi(x)$$

$$\hat{H}\,\psi(x) = E\,\psi(x)$$

EXPECTATION VALUE OF TOTAL ENERGY

$$\langle H \rangle = \int dx \;\; \psi^*(x)\;\hat{H}\;\psi(x)$$

$$= E\int dx \;\; \psi^*\psi$$

$$= E\underbrace{\int dx \;\; |\Psi|^2}_{1}$$

$$\underline{\underline{\langle H \rangle = E}}$$

↳ GENERAL SOLUTION : LINEAR COMBINATION OF SEPARABLE SOLUTIONS

$$\Psi_1(x,t) = \psi_1(x)\, e^{-\frac{i}{\hbar}E_1 t}$$

$$\Psi_2(x,t) = \psi_2(x)\, e^{-\frac{i}{\hbar}E_2 t}$$

....

$$\boxed{\Psi(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\psi_n(x)\, e^{-\frac{i}{\hbar}E_n t}}_{\Psi_n(x,t)}}$$

⇑

EACH SEPARABLE SOLUTION IS A STATIONARY STATE

↳ EXAMPLE : SUM OF 2 STATIONARY STATES

REAL

INITIAL STATE $(t=0)$ $\quad \Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x)$

↓

GENERAL STATE $\quad \Psi(x,t) = c_1\psi_1(x)\, e^{-\frac{i}{\hbar}E_1 t} + c_2\psi_2(x)\, e^{-\frac{i}{\hbar}E_2 t}$

PROBABILITY DENSITY $\quad |\Psi(x,t)|^2 = c_1^2\psi_1^2 + c_2^2\psi_2^2$

$$+ 2c_1c_2\psi_1\psi_2 \cos\left[(E_1 - E_2)\frac{t}{\hbar}\right]$$

OSCILLATION IN TIME OF $|\Psi|^2$ WITH FREQUENCY $(E_1 - E_2)/\hbar$

VISUALIZATION:

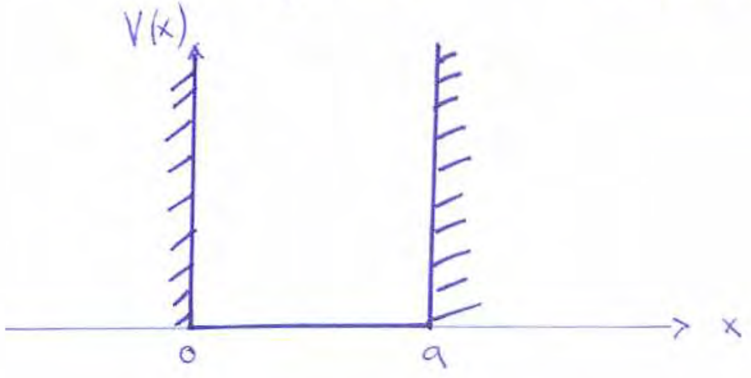
↳ SEE WEBPHYSICS.DAVIDSON.EDU/MJB/NCS_AAPT_QM_2002/WELCOME.HTML

MAX $\quad |\Psi|^2_{MAX} = (c_1\psi_1 + c_2\psi_2)^2$

MIN $\quad |\Psi|^2_{MIN} = (c_1\psi_1 - c_2\psi_2)^2$

⇒ 2.2 INFINITE SQUARE WELL

- $$V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & \text{OTHERWISE} \end{cases}$$



- FOR $x < 0$ OR $x > a$ $\quad \psi(x) = 0$

FOR $0 \leq x \leq a$

↳ $\psi(x)$ IS SOLUTION OF $-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi$

$$\downarrow \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\boxed{\frac{d^2\psi}{dx^2} = -k^2\psi}$$

GENERAL SOLUTION :

$$\psi(x) = A \sin kx + B \cos kx$$

A, B: REAL CONSTANTS.

- A & B DETERMINED BY <u>BOUNDARY CONDITIONS</u>

$$\psi(0) = \psi(a) = 0$$

ψ MUST BE A CONTINUOUS FUNCTION (IS ZERO OUTSIDE THE WELL)

$$\psi(0) = 0 \Rightarrow B = 0$$

$$\psi(a) = 0 \Rightarrow A \sin ka = 0$$

$$\Downarrow$$

$$ka = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

↳ $A = 0$ SOLUTION IS NOT ACCEPTABLE BECAUSE IT IS NOT NORMALIZABLE

↳ SAME APPLIES TO $k = 0$

↳ SOLUTIONS FOR $ka = -n\pi$

ARE EQUIVALENT TO THOSE FOR $ka = +n\pi$

BECAUSE $\sin(-x) = -\sin x$

& MINUS SIGN CAN BE ABSORBED IN A

↳ DISTINCT SOLUTIONS:

$$\boxed{k_n = n\frac{\pi}{a}} \quad , \quad \underline{\underline{n = 1, 2, 3, \ldots}}$$

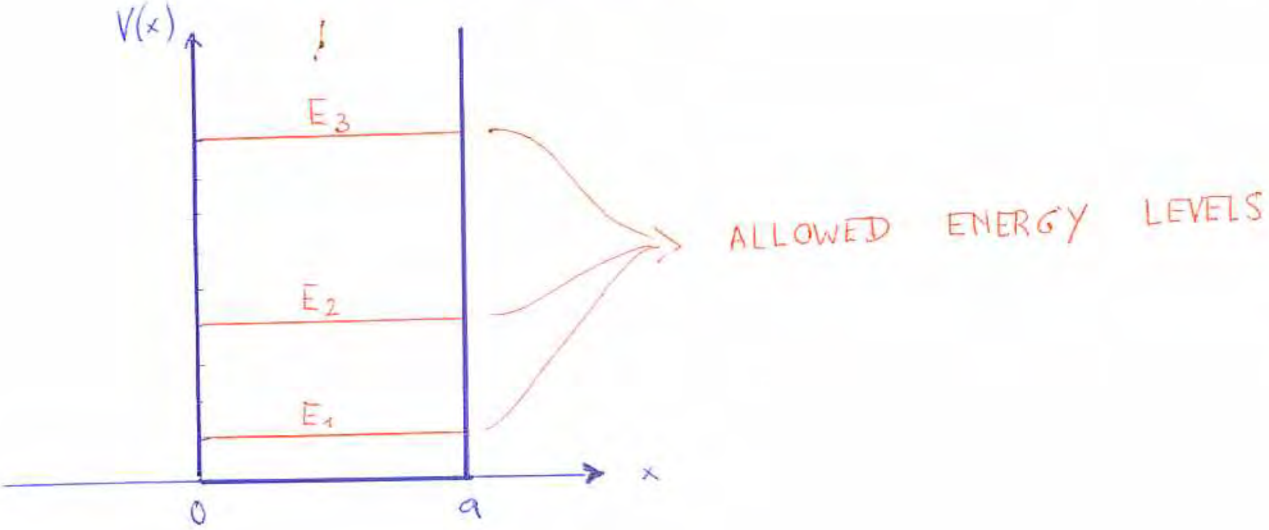
↳ BOUNDARY CONDITIONS DETERMINE

ALLOWED VALUES FOR k

" " " E

$$\boxed{E_n = \frac{\hbar^2 k_n^2}{2m} = n^2 \frac{\hbar^2 \pi^2}{2ma^2}} = n^2 E_1$$

ENERGY IS QUANTIZED



- NORMALIZATION

$$\int_{-\infty}^{+\infty} dx \; |\psi(x)|^2 = 1$$

$$\Downarrow$$

$$|A|^2 \int_0^a dx \; \sin^2\left(n \frac{\pi}{a} x\right) = 1$$

MATH HELP :

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

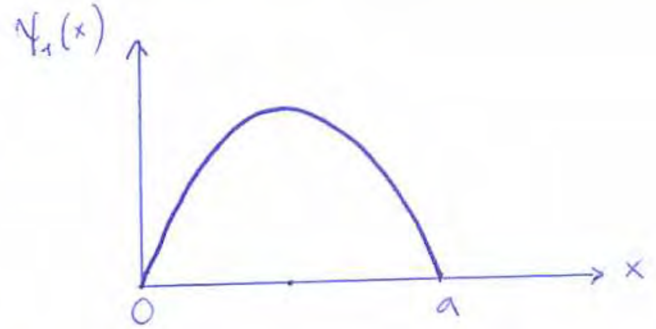
PROOF : SECOND TERM GIVES 0 UPON INTEGRATION

$$|A|^2 \cdot \frac{a}{2} = 1$$

$\boxed{A = \sqrt{\frac{2}{a}}}$ (ABSOLUTE PHASE HAS NO PHYSICAL SIGNIFICANCE)

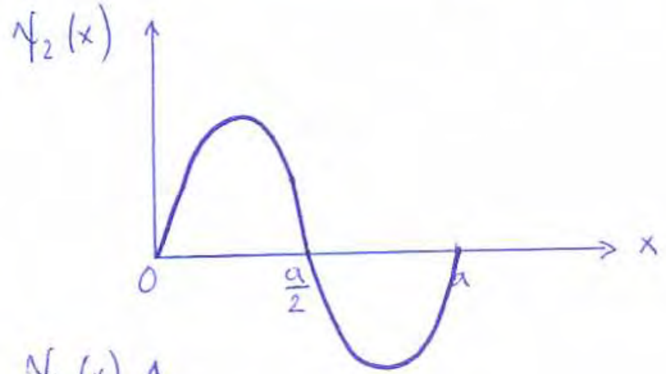
∴ SOLUTIONS

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)}$$



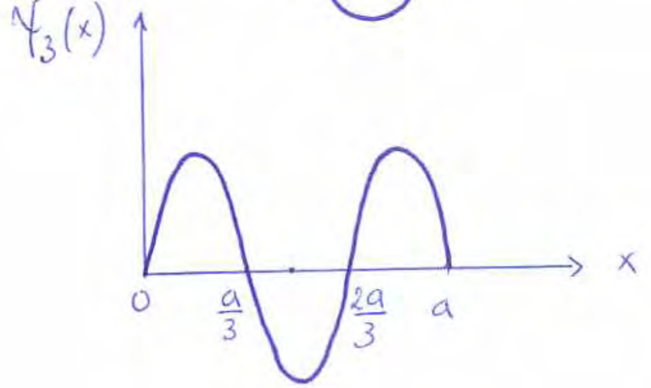
$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a} x\right)$$

GROUND STATE
(STATE OF LOWEST ENERGY)



$$\psi_2(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a} x\right)$$

1 NODE (ZERO) AT $x = \frac{a}{2}$



$$\psi_3(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi}{a} x\right)$$

2 NODES AT $x = \frac{a}{3}, \frac{2a}{3}$

$\psi_2, \psi_3, \ldots$ ARE CALLED EXCITED STATES

SOLUTIONS ARE STANDING WAVES

- PROPERTIES OF SOLUTIONS

↳ EVEN OR ODD w.r.t CENTER OF WELL

$n = 1, 3, 5, \ldots \Rightarrow$ EVEN (SYMMETRY PROPERTY)

$n = 2, 4, 6, \ldots \Rightarrow$ ODD

↳ EXCITED STATES HAVE NODES (ZEROS)

ψ_n HAS $n-1$ NODES.

THE HIGHER THE ENERGY, THE MORE NODES

↳ SOLUTIONS ARE ORTHOGONAL TO EACH OTHER

$$\boxed{\int dx \; \psi_m^*(x) \cdot \psi_n(x) = 0 \qquad n \neq m}$$

PROOF: $\int dx \; \psi_m^*(x)\, \psi_n(x)$

$$= \frac{2}{a}\int_0^a dx \; \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}x\right)$$

MATH HELP

$\sin x \cdot \sin y$

$= -\frac{1}{2}\left[\cos(x+y) - \cos(x-y)\right]$

$$= -\frac{1}{a}\int_0^a dx \left[\cos\left(\frac{(m+n)\pi}{a}x\right) - \cos\left(\frac{(m-n)\pi}{a}x\right)\right]$$

$$= -\frac{1}{a}\left[\frac{a}{\pi(m+n)}\sin\left(\frac{(m+n)\pi}{a}x\right)\right.$$

$$\left.-\frac{a}{\pi(m-n)}\sin\left(\frac{(m-n)\pi}{a}x\right)\right]_0^a$$

↳ ONLY VALID FOR $m \neq n$!

$$= -\frac{1}{\pi}\left\{\frac{\sin\left((m+n)\pi\right)}{(m+n)} - \frac{\sin\left((m-n)\pi\right)}{(m-n)}\right\}$$

$$\overset{!}{=} 0$$

↳ FOR $m = n$: NORMALIZATION CONDITION

∴ COMBINE ORTHOGONALITY & NORMALIZATION

INTO :

$$\boxed{\int dx\ \psi_m^*(x)\ \psi_n(x) = \delta_{nm}}$$

$$\delta_{nm} = \begin{cases} 1 & , \ n = m \\ 0 & , \ n \neq m \end{cases}$$

ψ_n's ARE CALLED <u>ORTHONORMAL</u>

↳ SOLUTIONS ARE COMPLETE

i.e. ANY OTHER FUNCTION CAN BE WRITTEN AS A LINEAR COMBINATION OF THEM

$$\boxed{f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)}$$

$$f(x) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a} x\right)$$

(FOURIER SERIES)

COEFFICIENTS c_n DETERMINED FROM ORTHONORMALITY :

$$\int dx\, \psi_m^*(x) f(x) = \sum_{n=1}^{\infty} c_n \underbrace{\int dx\, \psi_m^*(x) \psi_n(x)}_{\delta_{nm}}$$

$$= c_m$$

$$\boxed{c_n = \int dx\, \psi_n^*(x) f(x)}$$

↳ STATIONARY STATE

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar}E_n t}$$

WITH $E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$

↳ MOST GENERAL SOLUTION OF TIME DEPENDENT SCHRÖDINGER EQ. IN INFINITE SQUARE WELL POTENTIAL:

$$\boxed{\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar}E_n t}}$$

IF WE KNOW THE INITIAL $\Psi(x, t=0)$

WE CAN DETERMINE c_n

$\Downarrow$

WE ALSO KNOW SOLUTION AT ALL TIMES t

$$c_n = \int dx\, \psi_n^*(x)\, \Psi(x,0)$$

$$= \sqrt{\frac{2}{a}} \int_0^a dx \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)$$

$\therefore$ KNOWING $\Psi(x,t) \Rightarrow$ CALCULATE ALL OBSERVABLES

$\langle x \rangle, \langle p \rangle, \ldots$

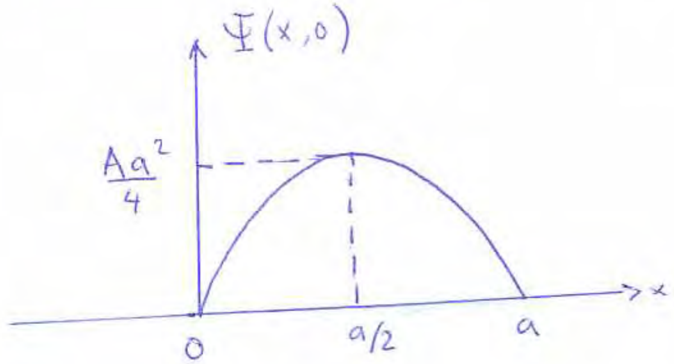
$\sigma_x, \sigma_y, \ldots$

- EXAMPLE

PARTICLE IN INFINITE SQUARE WELL

INITIAL W.F. $\Psi(x,0) = A\,x\,(a-x)$, $0 \leq x \leq a$

$= 0$, ELSEWHERE



DETERMINE W.F. AT ANY TIME

↳ NORMALIZATION

$$1 = \int dx \; |\Psi(x,0)|^2$$

$$= A^2 \int_0^a dx \; x^2 (a-x)^2 = A^2 \int_0^a dx \; x^2 \left(a^2 - 2ax + x^2\right)$$

$$= A^2 \left(\frac{a^5}{3} - \frac{2a^5}{4} + \frac{a^5}{5} \right)$$

$$= A^2 \frac{a^5}{30}$$

$$\boxed{A = \sqrt{\frac{30}{a^5}}}$$

$$\hookrightarrow \quad \Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar}E_n t}$$

$$c_n = \sqrt{\frac{2}{a}} \int_0^a dx \; \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)$$

$$c_n = \sqrt{\frac{2}{a}} \underset{\uparrow \atop \sqrt{\frac{30}{a^5}}}{A} \int_0^a dx \; \sin\left(\frac{n\pi}{a}x\right) \; x(a-x)$$

$$= \frac{2\sqrt{15}}{a^3} \left\{ a\int_0^a dx \;\; x \sin\left(\frac{n\pi}{a}x\right) - \int_0^a dx \;\; x^2 \sin\left(\frac{n\pi}{a}x\right) \right\}$$

$$= \frac{2\sqrt{15}}{a^3} \left\{ -\frac{a^2}{n\pi} \; x \cos\left(\frac{n\pi}{a}x\right)\Big|_0^a \right.$$

$$+ \cancel{\frac{a^2}{n\pi} \int_0^a dx \; \cos\left(\frac{n\pi}{a}x\right)} \quad 0$$

$$+ \frac{a}{n\pi} \; x^2 \cos\left(\frac{n\pi}{a}x\right)\Big|_0^a$$

$$\left. - \frac{2a}{n\pi} \int_0^a dx \; x \; \cos\left(\frac{n\pi}{a}x\right) \right\}$$

$$= \frac{2\sqrt{15}}{a^3} \left\{ \cancel{-\frac{a^3}{n\pi} \cos(n\pi) + \frac{a^3}{n\pi}\cos(n\pi)} \right.$$

$$- \frac{2a^2}{(n\pi)^2} \; \cancel{x \sin\frac{n\pi}{a}x \Big|_0^a} \; 0 + \frac{2a^2}{(n\pi)^2} \int_0^a dx \sin\left(\frac{n\pi}{a}x\right)$$

$$c_n = \frac{2\sqrt{15}}{a^3} \cdot (-1)\, \frac{2a^2}{(n\pi)^2} \cdot \frac{a}{n\pi} \quad \cos\left(\frac{n\pi}{a}x\right)\Big|_0^a$$

$$c_n = \frac{4\sqrt{15}}{(n\pi)^3} \left[1 - \cos(n\pi)\right]$$

$$\boxed{c_n = \begin{cases} 0 & , \quad n \text{ EVEN} \\ \dfrac{8\sqrt{15}}{(n\pi)^3} & , \quad n \text{ ODD} \end{cases}}$$

$$\boxed{\Psi(x,t) = \sqrt{\frac{30}{a}} \left(\frac{2}{\pi}\right)^3 \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar}E_n t}}$$

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$$

c_n : OVERLAP OF WAVEFUNCTION WITH n-th STATIONARY STATE

LARGEST OVERLAP WITH c_1

$$|c_1|^2 = \left(\frac{8\sqrt{15}}{\pi^3}\right)^2 = 0.99855\dots$$

$$(99.9\,\%)$$

$$c_n \sim \frac{1}{n^3}$$

↳ $|c_n|^2$ PROBABILITY TO FIND Ψ IN STATE n

$$\boxed{\sum_{n=1}^{\infty} |c_n|^2 = 1}$$

PROOF:

$$1 = \int dx \, |\Psi(x,0)|^2$$

$$= \int dx \left(\sum_{m=1}^{\infty} c_m \psi_m(x) \right)^* \left(\sum_{n=1}^{\infty} c_n \psi_n(x) \right)$$

$$= \sum_n \sum_m c_n c_m^* \underbrace{\int dx \, \psi_m^*(x) \psi_n(x)}_{\delta_{nm}}$$ ORTHONORMAL

$$= \sum_{n=1}^{\infty} |c_n|^2$$

■ QED

↳ EXPECTATION VALUE OF ENERGY

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$\langle H \rangle = \int dx \, \Psi \hat{H} \Psi$$

$$= \sum_n \sum_m c_m^* c_n \int dx \, \psi_m^*(x) \underbrace{\hat{H} \psi_n(x)}_{E_n \psi_n(x)}$$

$$= \sum_n \sum_m c_m^* c_n E_n \int dx \, \psi_m^*(x) \psi_n(x)$$

$$\boxed{\langle H \rangle = \sum_{n=1}^{\infty} E_n |c_n|^2}$$

- CHECK OF UNCERTAINTY PRINCIPLE FOR INFINITE SQUARE WELL

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2$$

FOR STATIONARY STATE $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$.

$$\hookrightarrow \langle x \rangle = \int_0^a dx \; \psi_n^*(x) \; x \; \psi_n(x)$$

$$= \frac{2}{a} \int_0^a dx \; x \; \underbrace{\sin^2\left(\frac{n\pi}{a} x\right)}_{\frac{1}{2}\left(1 - \cos\left(\frac{2n\pi}{a} x\right)\right)}$$

$$= \frac{2}{a} \int_0^a dx \; \frac{x}{2}$$

$$\underline{\underline{\langle x \rangle = \frac{1}{2} a}} \qquad \text{INDEPENDENT OF } n$$

$$\hookrightarrow \langle p \rangle = m \frac{d}{dt} \langle x \rangle = 0$$

$$\underset{\text{CHECK}}{=} m \int_0^a dx \; \psi_n^*(x) \left(-i\hbar \frac{d}{dx}\right) \psi_n(x)$$

$$= -i\hbar m \frac{2}{a} \cdot \frac{2n\pi}{a} \underbrace{\int_0^a dx \; \sin\left(\frac{n\pi}{a} x\right) \cdot \cos\left(\frac{n\pi}{a} x\right)}_{0}$$

$$\underline{\underline{\langle p \rangle = 0}}$$

$$\hookrightarrow \langle x^2 \rangle = \frac{2}{a} \int_0^a dx \; x^2 \underbrace{\sin^2\left(\frac{n\pi}{a} x\right)}_{\frac{1}{2}\left(1 - \cos\left(\frac{2n\pi}{a} x\right)\right)}$$

$$= \frac{1}{a} \left\{ \frac{a^3}{3} - \frac{a}{2n\pi} \underbrace{\cancel{x^2 \sin\left(\frac{2n\pi}{a} x\right)\Big|_0^a}}_{0} + \frac{a}{2n\pi} \cdot 2 \int_0^a dx \; x \sin\left(\frac{2n\pi}{a} x\right) \right\}$$

$$= \frac{1}{a} \left\{ \frac{a^3}{3} - \frac{a}{n\pi} \cdot \frac{a}{2n\pi} \; x \cos\left(\frac{2n\pi}{a} x\right) \Big|_0^a + \frac{a}{n\pi} \cdot \frac{a}{2n\pi} \underbrace{\cancel{\int_0^a dx \; \cos\left(\frac{2n\pi}{a} x\right)}}_{0} \right\}$$

$$= \frac{1}{a} \left\{ \frac{a^3}{3} - \frac{a^3}{2(n\pi)^2} \cdot \underbrace{\cos(2n\pi)}_{1} \right\}$$

$$\langle x^2 \rangle = a^2 \left[\frac{1}{3} - \frac{1}{2(n\pi)^2} \right]$$

$$\hookrightarrow \langle p^2 \rangle = \int dx \; \psi_n^*(x) \left(-\hbar^2 \frac{d^2}{dx^2} \right) \psi_n(x)$$

$$= \int_0^a dx \; \psi_n^*(x) \left(2m E_n \right) \psi_n(x)$$

$$= 2m E_n$$

$$\langle p^2 \rangle = \hbar^2 \left(\frac{n\pi}{a} \right)^2$$

$$\hookrightarrow \sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \frac{a^2}{4} \left[\frac{1}{3} - \frac{2}{(n\pi)^2} \right]$$

$$\sigma_x = \frac{a}{2} \sqrt{\frac{1}{3} - \frac{2}{(n\pi)^2}}$$

$$\hookrightarrow \sigma_p^2 = \langle p^2 \rangle - \underbrace{\cancel{\langle p \rangle^2}}_{0}$$

$$\sigma_p = \hbar \frac{n\pi}{a}$$

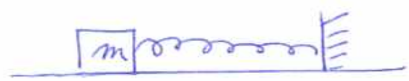
$$\hookrightarrow \boxed{\sigma_x \cdot \sigma_p = \frac{\hbar}{2} \cdot \underbrace{\sqrt{\frac{(n\pi)^2}{3} - 2}} \geqslant \frac{\hbar}{2}}$$

SMALLEST FOR $n = 1$: $\sqrt{\frac{\pi^2}{3} - 2} = 1.136$

$\Rightarrow$ 2.3 HARMONIC OSCILLATOR

- CLASSICAL H.O.

$F = -k\,x$



SPRING CONSTANT (k)

DISPLACEMENT (x)

$$m \frac{d^2 x}{dt^2} = -kx$$

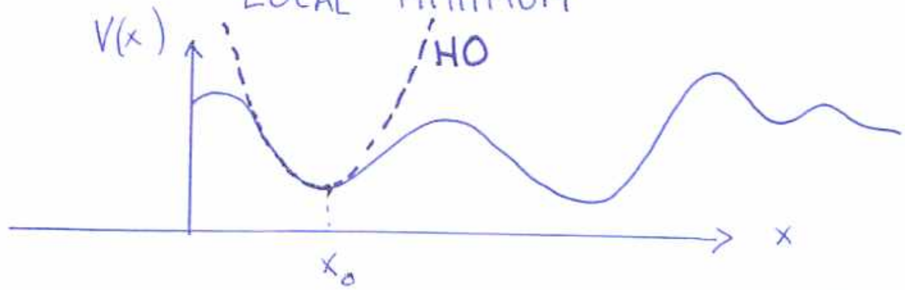
↳ SOLUTION: $x(t) = A \sin(\omega t) + B \cos(\omega t)$

$\omega = \sqrt{\frac{k}{m}}$ ANGULAR FREQUENCY

↳ POTENTIAL ENERGY $F = -\frac{dV}{dx}$

$$\underline{\underline{V(x) = \frac{1}{2} k x^2}}$$ PERFECT H.O.

↳ IN REALITY: H.O. IS GOOD APPROX. AROUND LOCAL MINIMUM



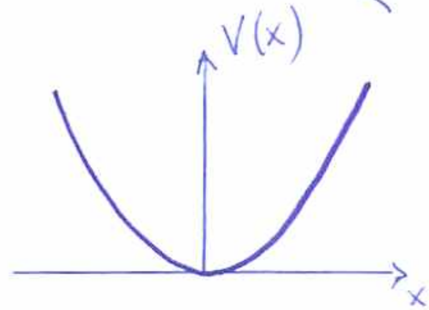
FOR X AROUND x_0: TAYLOR SERIES $V(x) = V(x_0) + \cancel{V'(x_0)}(x-x_0) + V''(x_0)\frac{1}{2}(x-x_0)^2 + \dots$

$V'(x_0) = 0$ LOCAL MIN.

H.O. $\boxed{V(x) - V(x_0) \simeq \frac{1}{2} V''(x_0)\,(x-x_0)^2} \rightarrow k \equiv V''(x_0).$

- QUANTUM H.O.

$$\boxed{V(x) = \frac{1}{2} m \omega^2 x^2}$$



TIME INDEPENDENT SCHRÖDINGER EQ.

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi(x) = E\,\psi(x)}$$

2 METHODS TO SOLVE ↗ ALGEBRAIC METHOD
↘ POWER SERIES METHOD

(1) ALGEBRAIC METHOD TO QUANTUM H.O.

$$\hookrightarrow \langle p \rangle = \int dx\; \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)$$

$$= \int dx\; \psi^*(x)\underbrace{\left(-i\hbar\frac{d}{dx}\right)}_{\hat{p}\,:\ \text{MOMENTUM OPERATOR}}\psi(x)$$

SCHRÖDINGER EQ.

$$\frac{\hat{p}^2}{2m}\psi + \frac{1}{2} m\omega^2 x^2 \psi = E\psi$$

$$\hat{H}\psi = E\psi$$

↳ HAMILTONIAN

$$\boxed{\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 x^2}$$

↳ IDEA TRY TO FACTOR $\hat{H}$

CONSIDER $\left(i\hat{p} + m\omega x\right)\cdot\left(-i\hat{p} + m\omega x\right)$

$$= \hat{p}^2 + i m\omega\left(\hat{p}x - x\hat{p}\right) + m^2\omega^2 x^2$$

$$= 2m\hat{H} + \underbrace{i m\omega\left(\hat{p}x - x\hat{p}\right)}_{\text{EXTRA TERM}}$$

ATTENTION $\hat{p}$ AND x DO NOT COMMUTE

COMMUTATOR $\boxed{\left[x, \hat{p}\right] \equiv x\hat{p} - \hat{p}x}$

$$\left[x, \hat{p}\right] f(x) = \left(-i\hbar\, x\frac{d}{dx} + i\hbar\frac{d}{dx}x\right) f(x)$$

$$= \cancel{-i\hbar\, x\frac{df}{dx}} + i\hbar f + \cancel{i\hbar\, x\frac{df}{dx}}$$

$$= i\hbar f(x)$$

$$\boxed{\left[x, \hat{p}\right] = i\hbar}$$

CANONICAL COMMUTATION RELATION

↳ DEFINE OPERATORS WHICH CORRESPOND WITH THE ABOVE FACTORS

i.e.

$$\boxed{\begin{aligned} a_+ &\equiv \frac{1}{\sqrt{\hbar\omega\, 2m}} \left(-i\hat{p} + m\omega x \right) \\ a_- &\equiv \frac{1}{\sqrt{\hbar\omega\, 2m}} \left(+i\hat{p} + m\omega x \right) \end{aligned}}$$

⇑

NORMALIZATION FACTOR TO GIVE SIMPLE PHYSICAL INTERPRETATION TO a_+ & a_- (SEE FURTHER ON)

↳ $$a_- a_+ = \frac{1}{\hbar\omega\, 2m} \left\{ 2m\hat{H} + i m\omega(-i\hbar) \right\}$$

$$\Downarrow$$

$$\hat{H} = \hbar\omega \left(a_- a_+ - \frac{1}{2} \right)$$

↳ ANALOGOUSLY WE CAN CALCULATE $a_+ a_-$

$$a_+ a_- = \frac{1}{\hbar\omega\, 2m} \left(-i\hat{p} + m\omega x \right)\left(+i\hat{p} + m\omega x \right)$$

$$= \frac{1}{\hbar\omega\, 2m} \left(2m\hat{H} + i m\omega \underbrace{[x, \hat{p}]}_{i\hbar} \right)$$

$$\hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$

$$a_- a_+ - \frac{1}{2} = a_+ a_- + \frac{1}{2}$$

$$\downarrow$$

$$\boxed{[a_-, a_+] = 1.}$$

↳ SCHRÖDINGER EQ.

$$\hat{H}\psi = E\psi$$

$$\boxed{\begin{aligned}\hat{H} &= \hbar\omega\left(a_+ a_- + \tfrac{1}{2}\right)\\ &= \hbar\omega\left(a_- a_+ - \tfrac{1}{2}\right)\end{aligned}}$$

$$\hbar\omega\left(a_+ a_- + \frac{1}{2}\right)\psi = E\psi$$

OR EQUIVALENTLY

$$\hbar\omega\left(a_- a_+ - \frac{1}{2}\right)\psi = E\psi$$

CRUCIAL STEP — IF ψ IS SOLUTION OF SCHRÖDINGER EQ. WITH ENERGY E

$$\Downarrow$$

$a_+\psi$ IS SOLUTION WITH ENERGY $E + \hbar\omega$

$a_-\psi$ IS SOLUTION WITH ENERGY $E - \hbar\omega$

PROOF $\quad \hat{H}\psi = E\psi$

(1) $$\hat{H}(a_+\psi) = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)(a_+\psi)$$

$$= \hbar\omega\left(a_+ \underbrace{a_- a_+} + \frac{1}{2}a_+\right)\psi$$

$$\downarrow \quad a_- a_+ = 1 + a_+ a_-$$

$$\hat{H}(a_+\psi) = \hbar\omega\left(a_+ + a_+a_+a_- + \tfrac{1}{2}a_+\right)\psi$$

$$= a_+ \quad \hbar\omega\left(a_+a_- + \tfrac{1}{2} + 1\right)\psi$$

$$\downarrow \quad \hat{H} = \hbar\omega\left(a_+a_- + \tfrac{1}{2}\right)$$

$$= a_+\left(\hat{H} + \hbar\omega\right)\psi$$

$$= a_+\left(E + \hbar\omega\right)\psi$$

$$= \left(E + \hbar\omega\right)\left(a_+\psi\right).$$

$\therefore$ $(a_+\psi)$ IS SOLUTION WITH ENERGY $E + \hbar\omega$

② $$\hat{H}(a_-\psi) = \hbar\omega\left(a_-a_+ - \tfrac{1}{2}\right)(a_-\psi)$$

$$= \hbar\omega\left(a_-a_+a_- - \tfrac{1}{2}a_-\right)\psi$$

$$= a_- \underbrace{\hbar\omega\left(a_+a_- - \tfrac{1}{2}\right)}_{\hat{H} - \hbar\omega}\psi$$

$$= a_-\left(\hat{H} - \hbar\omega\right)\psi$$

$$= a_-\left(E - \hbar\omega\right)\psi$$

$$= \left(E - \hbar\omega\right)\left(a_-\psi\right)$$

$\therefore$ $(a_-\psi)$ IS SOLUTION WITH ENERGY $E - \hbar\omega$

↳ BY APPLYING $a_{\pm}$ OPERATOR ON
A GIVEN SOLUTION WITH ENERGY E
ONE GENERATES A NEW SOLUTION WITH ENERGY $E \pm \hbar\omega$

$$a_{+} \leftrightarrow E + \hbar\omega$$
$$a_{-} \leftrightarrow E - \hbar\omega$$

a_{+} CALLED 'RAISING' OPERATOR
↳ RAISES ENERGY BY 'QUANTUM' $\hbar\omega$

a_{-} CALLED 'LOWERING' OPERATOR
↳ LOWERS ENERGY BY 'QUANTUM' $\hbar\omega$

- ONE CAN APPLY a_{+} SUCCESSIVELY TO GET STATES OF HIGHER ENERGY

e.g. $$\hat{H}\left(a_{+}^{2}\psi\right) = \left(E + 2\hbar\omega\right)\left(a_{+}^{2}\psi\right)$$
$$\vdots$$
$$\hat{H}\left(a_{+}^{n}\psi\right) = \left(E + n\hbar\omega\right)\left(a_{+}^{n}\psi\right)$$

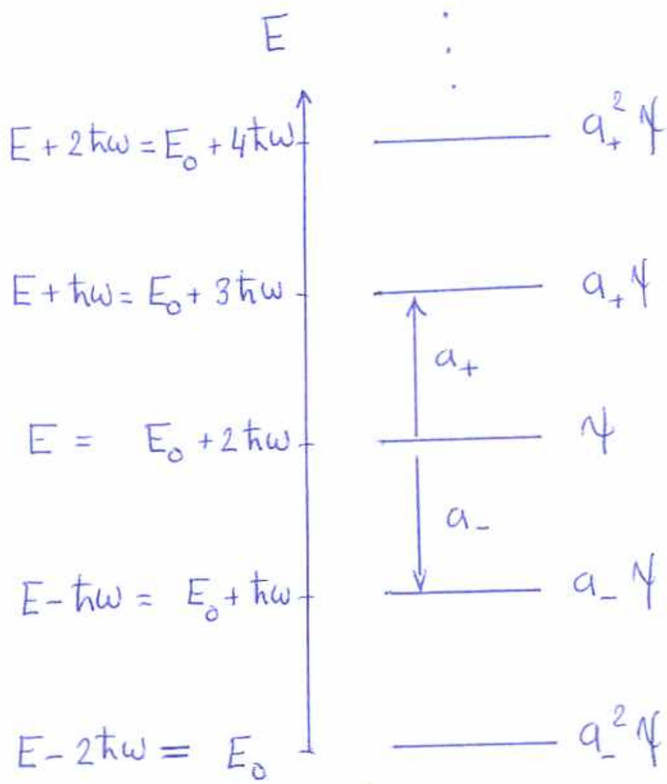
- ONE CAN APPLY a_{-} SUCCESSIVELY TO GET STATES OF LOWER ENERGY

e.g. $$\hat{H}\left(a_{-}^{2}\psi\right) = \left(E - 2\hbar\omega\right)\left(a_{-}^{2}\psi\right)$$
$$\vdots$$

BUT PAY ATTENTION !

BY REPEATED APPLICATION OF a_-
ONE WOULD EVENTUALLY REACH A STATE WITH NEGATIVE ENERGY
↓
UNPHYSICAL

∴ THERE MUST BE A STATE OF LOWEST ENERGY E_0
↳ GROUND STATE



ψ ⟹ LET'S SAY WE KNOW THIS SOLUTION

$a_-^2\psi \equiv \underbrace{\psi_0}$

GROUND STATE
= STATE WITH LOWEST ENERGY

FOR <u>GROUND STATE</u>

$$\underline{\underline{a_-\psi_0 = 0}}$$

i.e. ONE CANNOT GET A STATE OF LOWER ENERGY THAN ψ_0

$$\hat{H}(a_-\psi_0) = 0$$

<u>NOT</u> NORMALIZABLE SOLUTION OF SCHRÖDINGER EQ.

↳ USE CONDITION $\underline{\underline{a_- \psi_0 = 0}}$ TO DETERMINE

GROUND STATE ψ_0

$$a_- \psi_0 = 0$$

$$\Updownarrow$$

$$\frac{1}{\sqrt{\hbar\omega\, 2m}} \left(i\hat{p} + m\omega x \right) \psi_0 = 0$$

$$\left(\hbar \frac{d}{dx} + m\omega x \right) \psi_0 = 0$$

$$\boxed{\frac{d\psi_0}{dx} = - \frac{m\omega}{\hbar}\, x\, \psi_0}$$

SOLUTION : $\psi_0(x) = A\, e^{-\frac{m\omega}{2\hbar} x^2}$

USE NORMALIZATION TO DETERMINE A :

$$1 = \int_{-\infty}^{+\infty} dx\, |\psi_0(x)|^2 = |A|^2 \underbrace{\int_{-\infty}^{+\infty} dx\, e^{-\frac{m\omega}{\hbar} x^2}}$$

GAUSSIAN INTEGRAL (MATH HELP)

$$\int_{-\infty}^{+\infty} dx\, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$$

$$= |A|^2 \cdot \sqrt{\frac{\pi\hbar}{m\omega}}$$

$$A = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4}$$

∴ GROUND STATE OF QUANTUM H.O.

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}$$

↳ GROUND STATE ENERGY E_0

$$E_0 = \langle \hat{H} \rangle = \int_{-\infty}^{+\infty} dx \; \psi_0^*(x) \; \hat{H} \; \psi_0(x)$$

$$\hat{H} \downarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

EVALUATE ....

BUT THERE IS A SHORTER WAY TO DETERMINE E_0

$$\hat{H} = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)$$

$$\hat{H}\psi_0 = E_0\psi_0 = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)\psi_0$$

$\downarrow$ $a_-\psi_0 = 0$

THIS DEFINES THE GROUND STATE ψ_0.

$$= \frac{1}{2}\hbar\omega \; \psi_0$$

$$\boxed{E_0 = \frac{1}{2}\hbar\omega}$$

GROUND STATE ENERGY OF QUANTUM H.O.
IS NOT ZERO ! (IN CLASSICAL LIMIT $\hbar \to 0$, $E_0 \to 0$)

↳ ANY OTHER SOLUTION

BY REPEATED APPLICATION OF a_+ ON ψ_0.

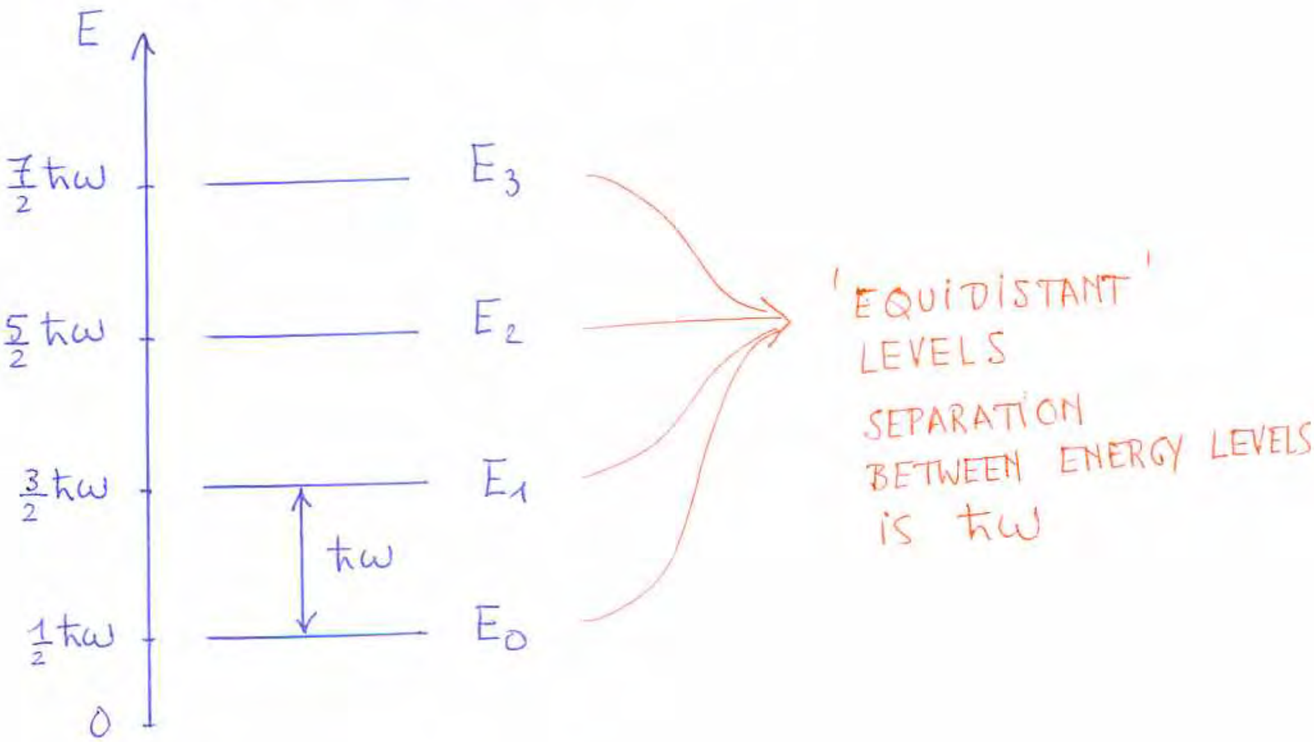
$$\boxed{\psi_n(x) = A_n \, a_+^n \, \psi_0}$$

↑
NORMALIZATION CONSTANT

ψ_n HAS ENERGY $E_n = E_0 + n\hbar\omega$

$$\boxed{E_n = \hbar\omega\left(\frac{1}{2} + n\right)}$$

SPECTRUM OF QUANTUM H.O.



e.g. FIRST EXCITED STATE

$$\psi_1(x) = A_1 \; a_+ \psi_0(x)$$

$$= A_1 \frac{1}{\sqrt{\hbar\omega 2m}} \left(-i\hat{p} + m\omega x\right) \underbrace{\psi_0(x)}_{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}$$

$$= A_1 \frac{1}{\sqrt{\hbar\omega 2m}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(-\hbar\frac{d}{dx} + m\omega x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= A_1 \frac{1}{\sqrt{\hbar\omega 2m}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega x \; e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\boxed{\psi_1(x) = A_1 \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} \; x \; e^{-\frac{m\omega}{2\hbar}x^2}}$$

NORMALIZATION $1 = \int_{-\infty}^{+\infty} dx \; |\psi_1(x)|^2$

$$= |A_1|^2 \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \frac{2m\omega}{\hbar} \cdot \underbrace{\int_{-\infty}^{+\infty} dx \; x^2 e^{-\frac{m\omega}{\hbar}x^2}}$$

MATH HELP : $\int_{-\infty}^{+\infty} dx \; x^2 e^{-\lambda x^2} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \cdot$

$\downarrow$

$$= |A_1|^2 \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \frac{2m\omega}{\hbar} \; \frac{1}{2} \left(\frac{\pi\hbar}{m\omega}\right)^{1/2} \frac{\hbar}{m\omega}$$

$$= |A_1|^2$$

$A_1 = 1.$

$$\hookrightarrow \boxed{A_n = \frac{1}{\sqrt{n!}}}$$

PROOF

$$\begin{array}{|l} a_+ \psi_n = c_n \psi_{n+1} \\ a_- \psi_n = d_n \psi_{n-1}. \end{array}$$

$$a_+ a_- \psi_n = d_n' \psi_n$$

WE KNOW $\begin{cases} \hat{H} \psi_n = E_n \psi_n = \hbar\omega \left(\frac{1}{2} + n\right) \psi_n \\ \hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2}\right) \end{cases}$

$$\boxed{a_+ a_- \psi_n = n \psi_n}$$

"NUMBER" OPERATOR COUNTS THE STATE n

$$\boxed{a_- a_+ \psi_n = \left(1 + a_+ a_-\right) \psi_n = (n+1) \psi_n}$$

NORMALIZATION: $$\int_{-\infty}^{+\infty} dx \underbrace{\left(a_+ \psi_n\right)^*}_{\frac{1}{\sqrt{\hbar\omega 2m}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\psi_n^*} \left(a_+ \psi_n\right) = |c_n|^2 \underbrace{\int_{-\infty}^{+\infty} dx \, |\psi_{n+1}|^2}_{1}$$

↓ INTEGRATION BY PARTS

$$|c_n|^2 = \int_{-\infty}^{+\infty} dx \; \psi_n^* \frac{1}{\sqrt{\hbar\omega 2m}} \left(+ \hbar \frac{d}{dx} + m\omega x \right) a_+ \psi_n$$

$$= \int_{-\infty}^{+\infty} dx \; \psi_n^* \; a_- a_+ \psi_n$$

$$= (n+1)$$

$$\therefore \quad \boxed{a_+ \psi_n = \sqrt{n+1}\, \psi_{n+1}}$$

ANALOGOUSLY $\quad |d_n|^2 = n$

$$\boxed{a_- \psi_n = \sqrt{n}\, \psi_{n-1}}$$

$$\leadsto \quad \psi_1 = a_+ \psi_0$$

$$\psi_2 = \frac{1}{\sqrt{2}} a_+ \psi_1 = \frac{1}{\sqrt{2}} a_+^2 \psi_0$$

$$\psi_3 = \frac{1}{\sqrt{3}} a_+ \psi_2 = \frac{1}{\sqrt{3.2}} a_+^3 \psi_0 = \frac{1}{\sqrt{3!}} a_+^3 \psi_0$$

$$\vdots$$

$$\psi_n = \frac{1}{\sqrt{n}} a_+ \psi_{n-1} = \frac{1}{\sqrt{n(n-1)}} a_+^2 \psi_{n-2} = \frac{1}{\sqrt{n!}} a_+^n \psi_0$$

$$\Uparrow$$

$$A_n = \frac{1}{\sqrt{n!}}$$

■ QED

$\hookrightarrow$ SOLUTIONS OF H.O. ARE ORTHONORMAL

$$\boxed{\int_{-\infty}^{+\infty} dx \; \psi_m^*(x)\, \psi_n(x) = \delta_{nm}}$$

PROOF

$$\int_{-\infty}^{+\infty} dx \; \psi_m^*(x)\; a_+ a_- \psi_n(x)$$

BY PARTIAL INTEGRATION TWICE

$$= n \int_{-\infty}^{+\infty} dx \; \psi_m^*(x)\, \psi_n(x)$$

$$= \int_{-\infty}^{+\infty} dx \; \underbrace{\left(a_+ a_- \psi_m \right)^*}_{m\,\psi_m^*} \; \psi_n$$

$$= m \int_{-\infty}^{+\infty} dx \; \psi_m^* \psi_n$$

FOR $n \neq m \Rightarrow \int_{-\infty}^{+\infty} dx \; \psi_m^* \psi_n = 0.$

$n = m \Rightarrow \int_{-\infty}^{+\infty} dx \; |\psi_n|^2 = 1$ NORMALIZATION

$\blacksquare$ QED

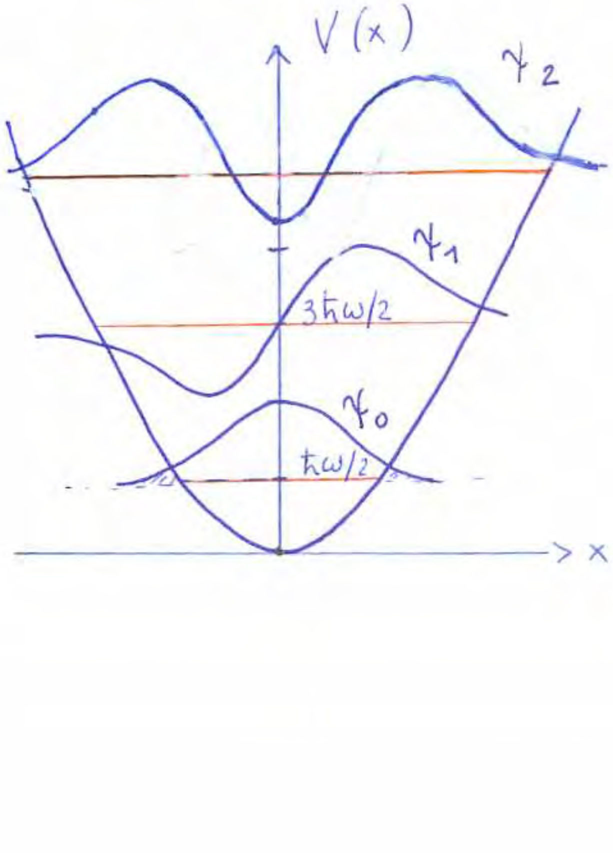
↳ PLOT OF LOWEST FEW WAVE FUNCTIONS.

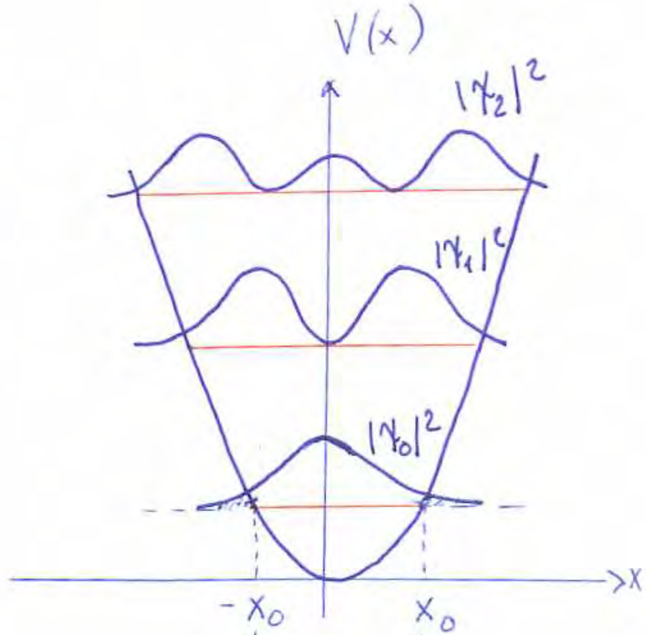
- $\psi_0(x) \sim e^{-\frac{m\omega}{2\hbar}x^2}$ EVEN IN x

$|\psi_0|^2 \sim e^{-\frac{m\omega}{\hbar}x^2}$

- $\psi_1 \sim x\, e^{-\frac{m\omega}{2\hbar}x^2}$ ODD IN x

$|\psi_1|^2 \sim x^2 e^{-\frac{m\omega}{2\hbar}x^2}$





CLASSICAL TURNING POINTS

$$V(x_0) = \frac{1}{2} m\omega^2 x_0^2$$

$$= E_0 = \frac{1}{2}\hbar\omega$$

$$\Downarrow$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

- FOR ENERGY LEVEL E_n

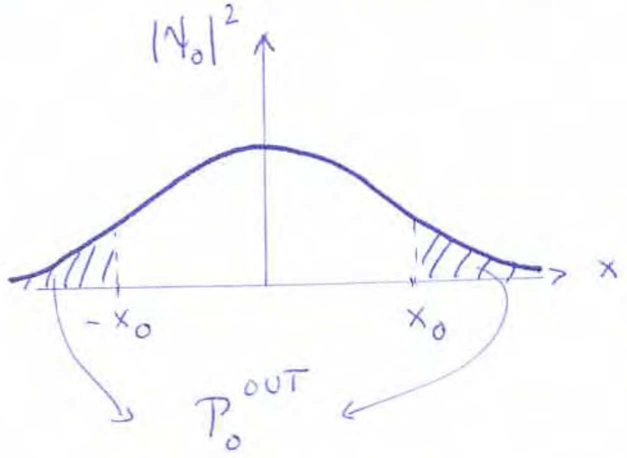
THE CLASSICAL TURNING POINTS x_n ARE

$$V(x_n) = \frac{1}{2} m \omega^2 x_n^2 = E_n$$

$$x_n = \sqrt{\frac{2E_n}{m\omega^2}}$$

- IN Q.M. THERE IS A NON-ZERO PROBABILITY TO FIND PARTICLE OUTSIDE THE CLASSICALLY ALLOWED REGION

FOR ψ_0 :



$$P_0^{OUT} = 2 \int_{x_0}^{\infty} dx \; |\psi_0(x)|^2$$

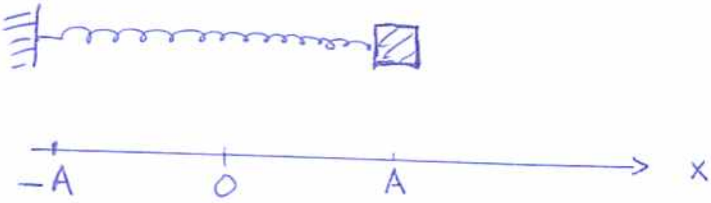
$$= 2 \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{\sqrt{\frac{\hbar}{m\omega}}}^{\infty} dx \; e^{-\frac{m\omega}{\hbar}x^2}$$

$\downarrow$ DIMENSIONLESS VAR $\xi \equiv x \Big/ \left(\frac{\hbar}{m\omega}\right)^{1/2}$

$$= \frac{2}{\sqrt{\pi}} \int_{1}^{\infty} d\xi \; e^{-\xi^2} = 0.1573 \quad 15.7\,\% \; !$$

CLASSICAL ↔ QUANTUM PROBABILITY DENSITIES FOR H.O.

- CLASSICAL



$$x(t) = A\cos(\omega t) \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\downarrow$$

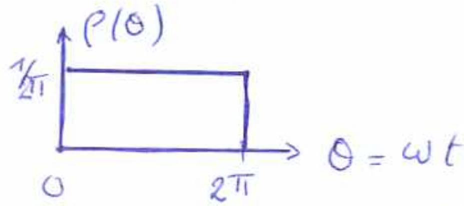
PERIOD $T = \dfrac{2\pi}{\omega}$

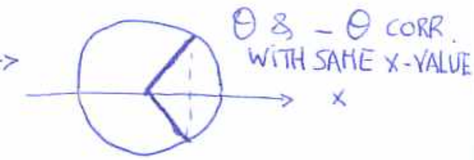
FOR $t \gg T$: H.O. GOES BACK & FORTH MANY, MANY TIMES

$P_{cl}(x)\,dx$: PROBABILITY TO FIND PARTICLE (▨ IN FIGURE) BETWEEN x & $x + dx$

↳ FOR $t \gg T$: EACH VALUE OF $\theta = \omega t$ WILL HAVE SAME PROBABILITY

$$\rho(\theta) = \frac{1}{2\pi}$$





$$P_{cl}(x)\,dx = 2\,\rho(\theta)\,d\theta$$

$$\left|\frac{dx}{d\theta}\right| = A\,\sin(\omega t) = A\sqrt{1 - \frac{x^2}{A^2}}$$

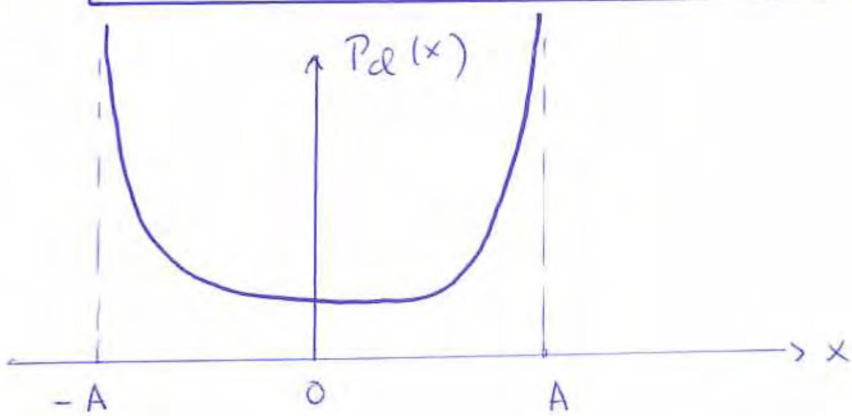
$$P_{cl}(x) = \frac{2}{2\pi} \left| \frac{d\theta}{dx} \right|$$

$$= \frac{2\omega}{2\pi} \cdot \left| \frac{dt}{dx} \right|$$

$$= \frac{2}{T} \cdot \frac{1}{|v|}$$

↳ VELOCITY

$$\boxed{P_{cl}(x) = \frac{1}{\pi} \frac{1}{A} \frac{1}{\sqrt{1 - \frac{x^2}{A^2}}}}$$



- QUANTUM

HOW DOES THIS PROBABILITY DENSITY COMPARE WITH QUANTUM?

$$P_{QM}(x) = |\Psi(x,t)|^2$$

IF H.O. IS IN STATIONARY STATE Ψ_n

WITH ENERGY $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

$$\boxed{P_{QM} = |\Psi_n(x)|^2}$$

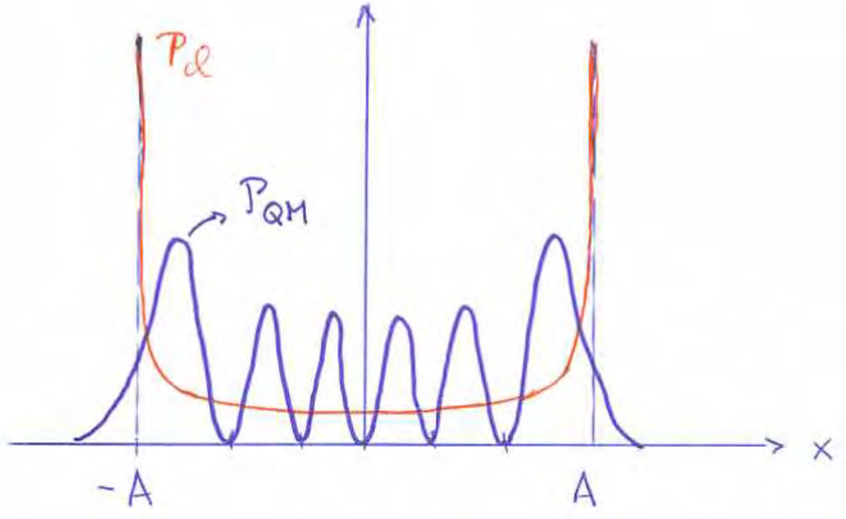
FOR ENERGY E_n : CLASSICAL TURNING POINTS A :

$$\frac{1}{2} m\omega^2 A^2 = \frac{\hbar\omega}{2}(2n+1)$$

$$A = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2n+1}$$

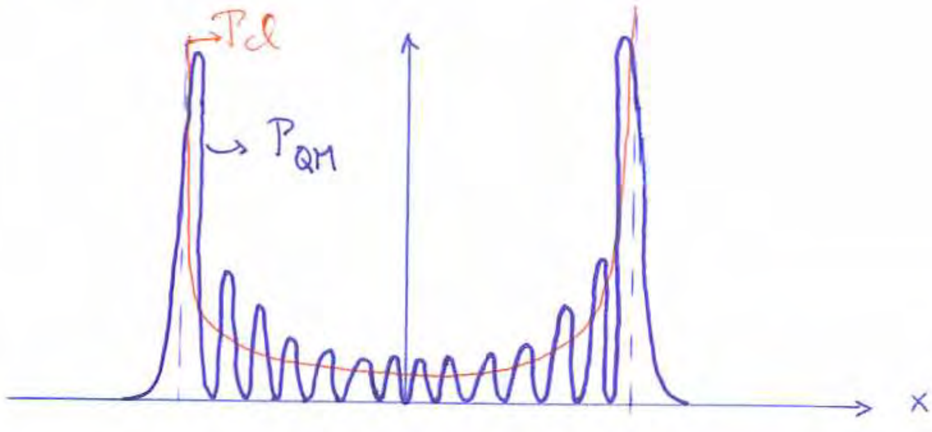
$$\boxed{P_{CL}(x) = \frac{1}{2\pi}\frac{1}{\sqrt{\hbar/m\omega}}\frac{1}{\sqrt{2n+1}}\cdot\frac{1}{\sqrt{1 - \frac{x^2}{(\hbar/m\omega)}\frac{1}{2n+1}}}}$$

GRAPHICAL e.g. $n = 5$ (5 NODES)



$$n=5: \qquad A = \sqrt{\frac{\hbar}{m\omega}}\ \underbrace{\sqrt{11}}_{3.32}$$

FOR $n \gg$



! QUANTUM PROBABILITY OSCILLATES AROUND CLASSICAL PROBABILITY

↳ CORRESPONDENCE PRINCIPLE

- VISUALIZATION

SEE WEBPHYSICS.DAVIDSON.EDU/MJB/NCS_AAPT_QM_2002/WELCOME.HTML

⇒ 2.4 THE FREE PARTICLE

- $V(x) = 0$ EVERY WHERE

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\Downarrow$$

$$\boxed{\frac{d^2\psi}{dx^2} = -k^2\psi} \qquad \boxed{E \equiv \frac{\hbar^2 k^2}{2m}}$$

k : WAVEVECTOR

↳ GENERAL SOLUTION $\psi(x) = A e^{ikx} + B e^{-ikx}$

SIMILAR AS INSIDE REGION OF INFINITE SQUARE WELL

DIFFERENCE : NO BOUNDARY CONDITIONS TO CONSTRAIN k

↓

FREE PARTICLE CAN MOVE WITH ANY VALUE OF E

↳ TIME DEPENDENCE

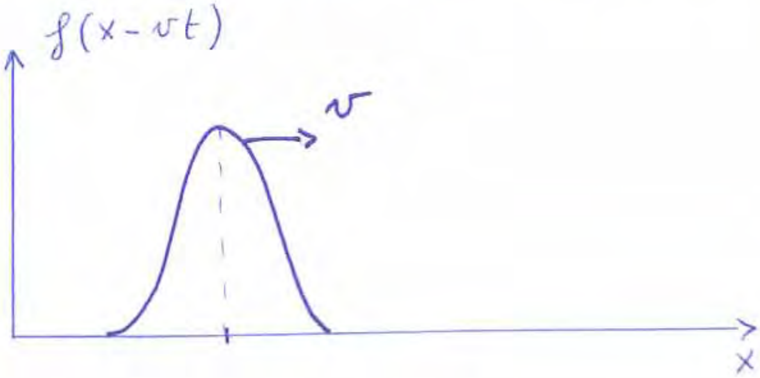
$$\Psi_k(x,t) = A\, e^{-\frac{i}{\hbar}Et}\, e^{ikx} + B\, e^{-\frac{i}{\hbar}Et}\, e^{-ikx}$$

$$= A\, e^{+ik\left(x - \frac{\hbar k}{2m}t\right)} + B\, e^{-ik\left(x + \frac{\hbar k}{2m}t\right)}$$

↳ SOLUTION DEPENDS ON CONTINUOUS VARIABLE k

STATIONARY STATE

$f(x \mp vt)$ CORRESPONDS WITH WAVE TRAVELLING WITH VELOCITY v IN $\pm$ X-DIRECTION



$$k = \pm \frac{\sqrt{2mE}}{\hbar}$$

$k > 0$: WAVE TRAVELING TO RIGHT

$k < 0$: WAVE TRAVELING TO LEFT

↳ MOMENTUM CARRIED BY WAVE

$$p = \hbar k \qquad k = \frac{2\pi}{\lambda} \Rightarrow p = \frac{h}{\lambda}$$

DE BROGLIE FORMULA

↳ SPEED OF WAVE

QUANTUM : $$v_{QM} = \frac{\hbar k}{2m} = \sqrt{\frac{E}{2m}}$$

CLASSICAL PARTICLE WITH KIN. ENERGY E

$$E = \frac{1}{2} m v^2 \Rightarrow v_{CL} = \sqrt{\frac{2E}{m}} = 2\, v_{QM}$$

! QUANTUM WAVE TRAVELS AT HALF THE SPEED OF CLASSICAL PARTICLE ? (SEE LATER)

↳ NORMALIZATION

$$\Psi_k(x,t) = A\, e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}$$

$$\int_{-\infty}^{+\infty} dx\, |\Psi(x,t)|^2 = |A|^2 \underbrace{\int_{-\infty}^{+\infty} dx \cdot 1}_{\infty}$$

NOT NORMALIZABLE !

UNPHYSICAL ⇒ A FREE PARTICLE CANNOT EXIST IN A STATIONARY STATE

BUT : WE CAN CONSTRUCT NORMALIZABLE SOLUTIONS AS LINEAR COMBINATIONS OF STATIONARY STATES

BECAUSE STATIONARY STATE DEPENDS ON CONTINUOUS VARIABLE k

GENERAL SOLUTION IS INTEGRAL OVER k.
(INSTEAD OF A SUM OVER A DISCRETE INDEX n)

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\; \phi(k)\; e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}}$$

⇒ WAVE PACKET

↑
CONVENIENT NORMALIZATION FACTOR

ROLE OF C_n BEFORE $\longleftrightarrow$ $\frac{1}{\sqrt{2\pi}}\,\phi(k)$

↳ HOW TO DETERMINE $\phi(k)$

USUALLY ONE IS GIVEN AN INITIAL WAVEFUNCTION

$\Psi(x, t=0)$ ⇒ FROM WHICH ONE DETERMINES $\phi(k)$

$$\boxed{\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\ \phi(k)\, e^{ikx}}$$

NORMALIZED

⇓

$\phi(k)$ DETERMINED BY FOURIER TRANSFORM

$$\boxed{\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\ e^{-ikx}\, \Psi(x,0)}$$

GENERAL SOLUTION OBTAINED BY PLUGGING $\phi(k)$ INTO $\Psi(x,t)$

FOURIER TRANSFORM

PROOF OF :

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{ikx}\, F(k) \\ &\Updownarrow \\ F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{-ikx}\, f(x) \end{aligned}}$$

↳ CONSIDER FIRST FINITE INTERVAL $x \in [-a, a]$

AND LET $a \to \infty$ AT END

GENERAL SOLUTION $(t=0)$

$$f(x) = \sum_{n=0}^{\infty} \left\{ a_n \sin\left(\frac{n\pi}{a} x\right) + b_n \cos\left(\frac{n\pi}{a} x\right) \right\}$$

PERIODIC BOUNDARY CONDITION $f(a) = f(-a)$

$$\cos(kx) = \cos(k(x+2a))$$

$$\sin(kx) = \sin(k(x+2a))$$

$$2ka = n \cdot 2\pi$$

$$k = \frac{n\pi}{a}$$

USING $e^{\pm ikx} = \cos kx \pm i \sin kx$

$$\cos\left(\frac{n\pi}{a}x\right) = \frac{1}{2}\left(e^{i\frac{n\pi}{a}x} + e^{-i\frac{n\pi}{a}x}\right)$$

$$\sin\left(\frac{n\pi}{a}x\right) = \frac{1}{2i}\left(e^{i\frac{n\pi}{a}x} - e^{-i\frac{n\pi}{a}x}\right)$$

$$f(x) = \sum_{n=0}^{\infty}\left\{\frac{1}{2}\left(-ia_n + b_n\right) e^{i\frac{n\pi}{a}x} + \frac{1}{2}\left(+ia_n + b_n\right) e^{-i\frac{n\pi}{a}x}\right\}$$

DEFINE
$$C_n \equiv \begin{cases} b_0 \, , & n = 0 \\ \frac{1}{2}\left(-ia_n + b_n\right), & n = 1, 2, 3, \dots \\ \frac{1}{2}\left(ia_{-n} + b_{-n}\right), & n = -1, -2, -3, \dots \end{cases}$$

$$\boxed{f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{i\frac{n\pi}{a}x}}$$

$$\hookrightarrow \boxed{c_n = \frac{1}{2a}\int_{-a}^{a} dx \; e^{-i\frac{n\pi}{a}x} f(x)}$$

PROOF:

$$\int_{-a}^{a} dx \; e^{-i\frac{n\pi}{a}x} f(x)$$

$$= \sum_{m=-\infty}^{+\infty} c_m \int_{-a}^{a} dx \; e^{-i\frac{n\pi}{a}x} \, e^{i\frac{m\pi}{a}x}$$

$$= \sum_{m=-\infty}^{+\infty} c_m \underbrace{\int_{-a}^{a} dx \; e^{-i\frac{\pi}{a}(n-m)x}}$$

FOR $n = m \Rightarrow +2a$.

FOR $n \neq m$ $\quad -\frac{a}{i\pi}\frac{1}{(n-m)} \, e^{-i\frac{\pi}{a}(n-m)x}\Big|_{-a}^{a}$

$$= \frac{ia}{\pi}\frac{1}{(n-m)} \underbrace{\left\{ e^{-i\pi(n-m)} - e^{i\pi(n-m)} \right\}}_{-2i\sin((n-m)\pi)}$$

$$-2i\sin((n-m)\pi) = 0$$

FOR $n \neq m$

$$= \sum_{m=-\infty}^{+\infty} c_m \cdot \delta_{nm}(2a)$$

$$= 2a\, c_n$$ ∎ QED

↳ EXTEND INTERVAL TO $[-\infty, +\infty]$

$a \to \infty$

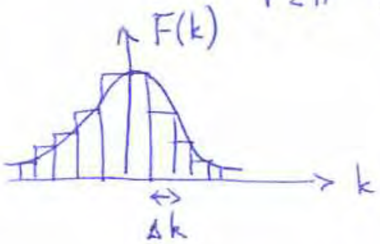
DEFINE $\underbrace{k_n = \frac{n\pi}{a}}$

$\Delta k = \frac{\pi}{a}$ INCREMENT IN k

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{i k_n x}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \frac{\sqrt{2\pi}\, a}{\pi} c_n \, . \, \Delta k \, . \, e^{i k_n x}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \underbrace{\left(\sqrt{\frac{2}{\pi}}\, a\, c_n \right)}_{\equiv\, F(k_n)} . \, \Delta k \;\, e^{i k_n x}$$



$$\xrightarrow[\Delta k \to 0]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \;\; F(k) \; e^{ikx}$$

$a \to \infty$

DEFINITION OF INTEGRAL.

$$\therefore \; F(k) = \sqrt{\frac{2}{\pi}}\, a\, c_n = \sqrt{\frac{2}{\pi}}\; a \, . \, \frac{1}{2a} \int_{-a}^{a} dx \;\, f(x)\, e^{-ikx}$$

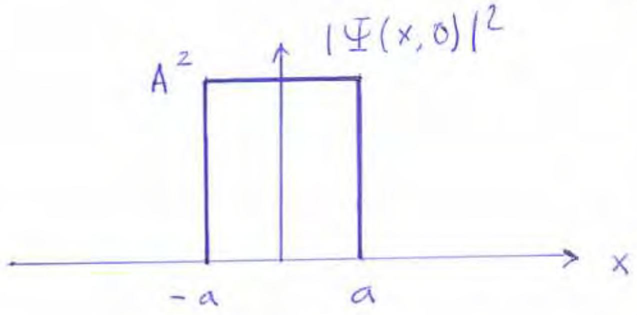
$$\xrightarrow[a \to \infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \;\; e^{-ikx} \;\, f(x)$$

∎ QED

• EXAMPLE

FREE PARTICLE INITIALLY ($t=0$) LOCALIZED IN INTERVAL $x \in [-a, a]$

$$\Psi(x,0) = \begin{cases} A & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$



WHAT IS $\Psi(x,t)$?

↳ NORMALIZATION $\displaystyle\int_{-\infty}^{+\infty} dx\, |\Psi(x,0)|^2 = 1$

$$\downarrow$$

$$2a\, A^2 = 1 \Rightarrow \boxed{A = \frac{1}{\sqrt{2a}}}$$

↳ $$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \phi(k)\, e^{ikx}\, e^{-i\frac{\hbar k^2}{2m}t}$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{-ikx}\, \Psi(x,0)$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dx \, \frac{1}{\sqrt{2a}} \, e^{-ikx}$$

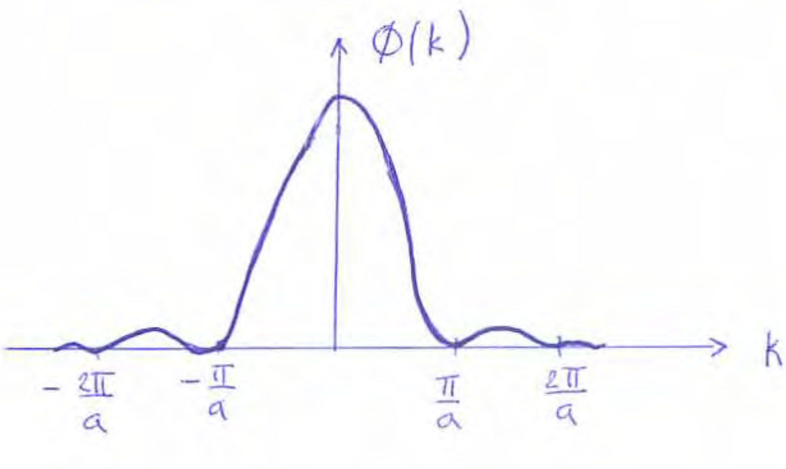
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \frac{1}{(-ik)} \, e^{-ikx} \Big|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \frac{1}{k} \underbrace{\frac{e^{ika} - e^{-ika}}{i}}_{2 \sin ka}$$

$$\boxed{\phi(k) = \frac{1}{\sqrt{\pi a}} \frac{\sin(ka)}{k}}$$

$\frac{\sin z}{z}$ HAS MAXIMUM AT $z = 0$

HAS ZEROS AT $z = \pm\pi, \pm 2\pi, \ldots \Rightarrow$ $k = \pm \frac{\pi}{a}, \pm\frac{2\pi}{a}, \ldots$

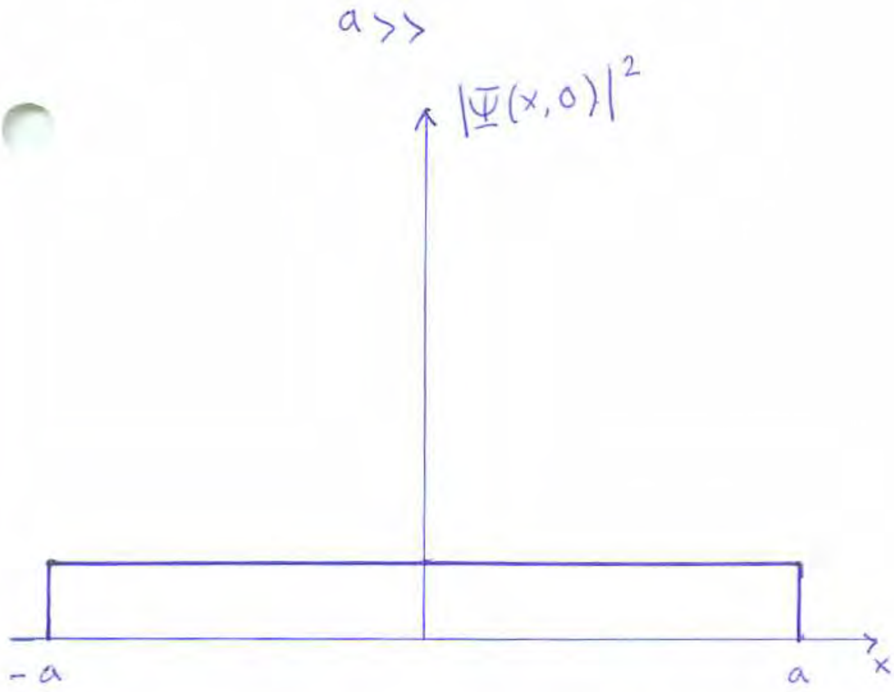


FOR $z \to 0$

$\frac{\sin z}{z} \to 1$

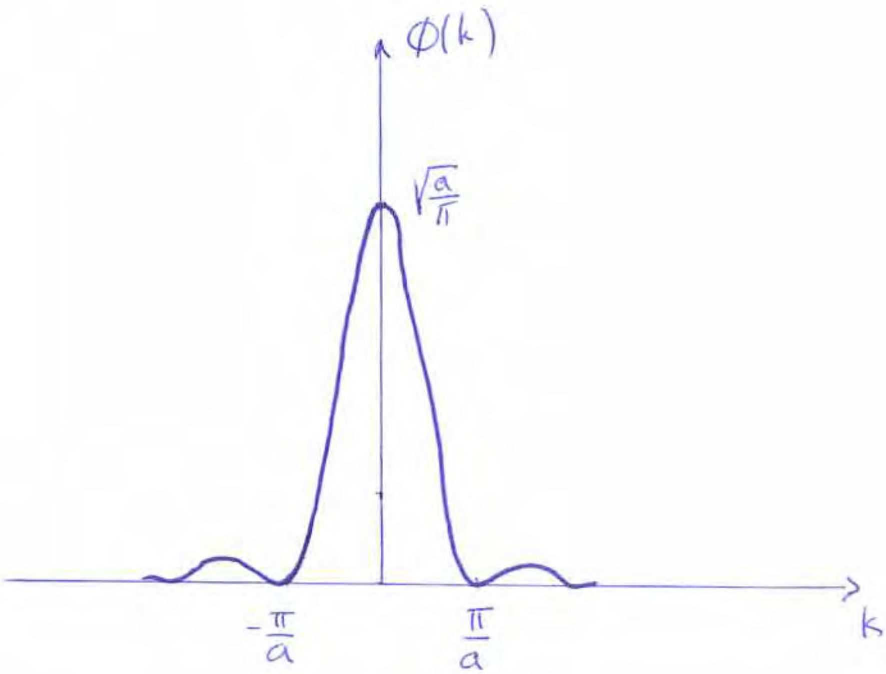
$\Downarrow$

$\phi(k = 0) = \sqrt{\frac{a}{\pi}}$

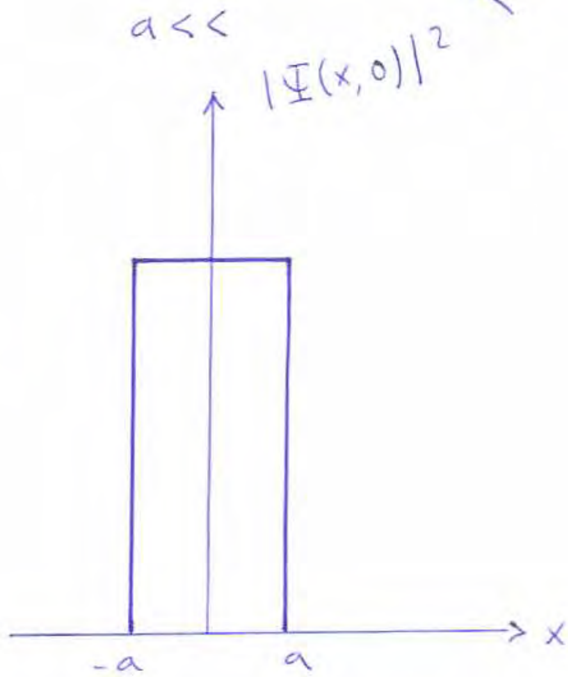


NOT WELL LOCALIZED
IN POSITION

$\Downarrow$

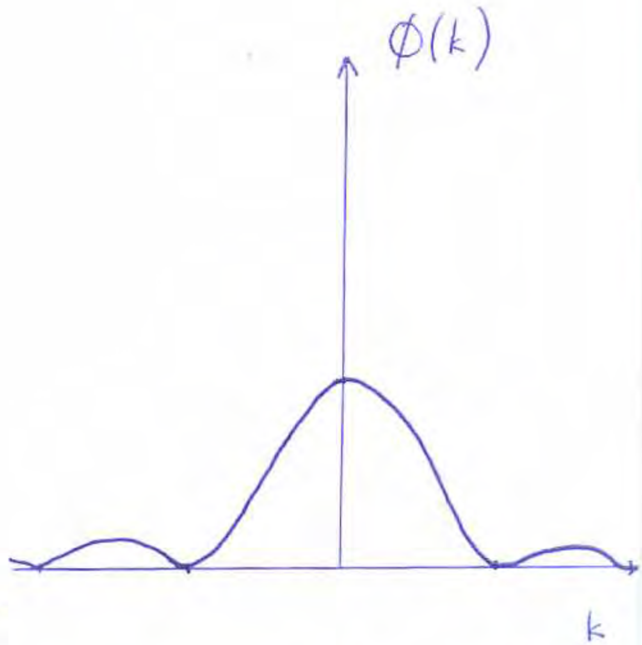


WELL LOCALIZED
IN MOMENTUM



WELL LOCALIZED
IN POSITION

$\Downarrow$



NOT WELL LOCALIZED
IN MOMENTUM

$\therefore$ HEISENBERG'S UNCERTAINTY PRINCIPLE

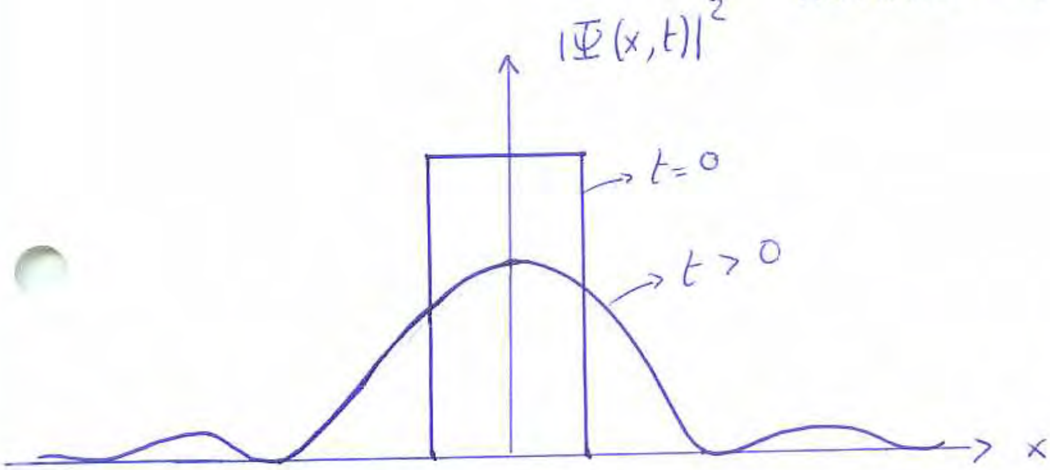
$\hookrightarrow$ PLUG $\phi(k)$ INTO $\Psi(x,t)$

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \; \phi(k) \; e^{ikx} \; e^{-i\frac{\hbar k^2}{2m}t}}$$

$$= \frac{1}{\sqrt{2a}} \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \; \frac{\sin(ka)}{k} \; e^{ikx} \; e^{-i\frac{\hbar k^2}{2m}t}$$

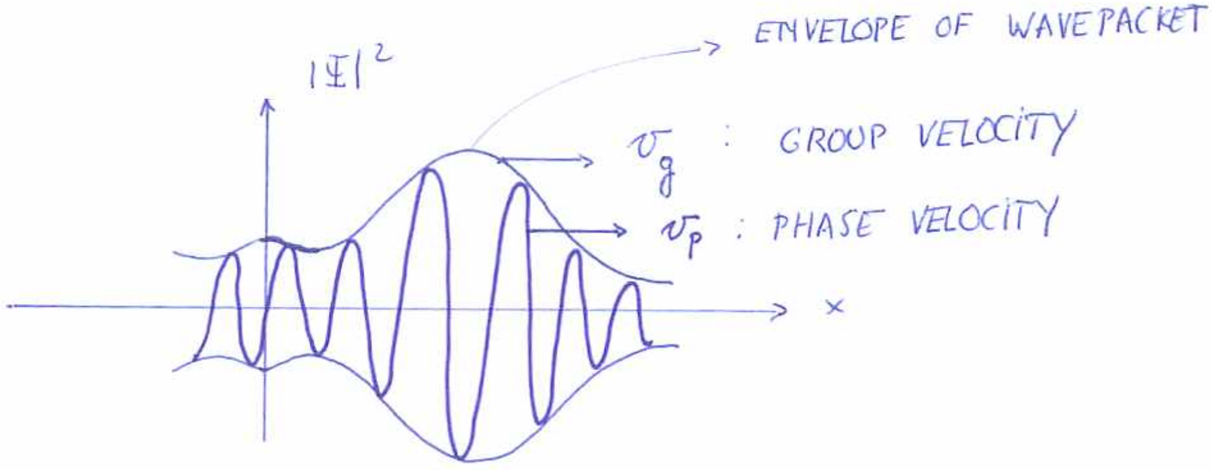
$\downarrow$

INTEGRAL CAN ONLY BE SOLVED NUMERICALLY



FOR $t>0$: WAVE PACKET SPREADS OUT

• GROUP VELOCITY <-> PHASE VELOCITY



$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \; \phi(k) \, e^{i(kx - \omega t)}$$

FOR FREE PARTICLE $\omega(k) = \frac{\hbar k^2}{2m}$

ASSUME $\phi(k)$ NARROWLY DISTRIBUTED AROUND k_0

TAYLOR EXPAND $\omega(k) \simeq \overbrace{\omega(k_0)}^{\omega_0 \, \equiv} + \omega_0'(k - k_0)$

+ CHANGE OF VARIABLES $k' = k - k_0$

$$\Psi(x,t) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk' \; \phi(k' + k_0) \, e^{i\left[(k_0 + k')x - (\omega_0 + \omega_0' k')t\right]}$$

$$= e^{i(-\omega_0 t + \omega_0' k_0 t)} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk' \; \phi(k' + k_0) \, e^{i(k_0 + k')(x - \omega_0' t)}$$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk' \, \phi(k' + k_0) \, e^{i(k_0+k')x}$$

SHIFT FROM $x \to x - \omega_0' t$

$$\boxed{\Psi(x,t) = e^{-i(\omega_0 - \omega_0' k_0)t} \, \Psi(x - \omega_0' t, 0)}$$

↳ WAVE PACKET MOVES AT SPEED

$$v_g = \frac{d\omega}{dk} = \omega_0' \quad : \text{GROUP VELOCITY}$$

↳ PHASE VELOCITY

$$v_p = \frac{\omega}{k}$$

FOR Q.M FREE PARTICLE $v_g = 2 v_p$

↑

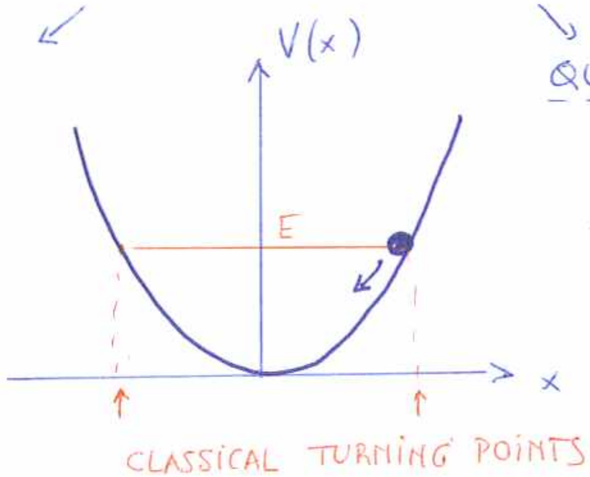
CLASSICAL VELOCITY

⇒ 2.5 DELTA - FUNCTION POTENTIAL

- **BOUND STATES vs SCATTERING STATES**

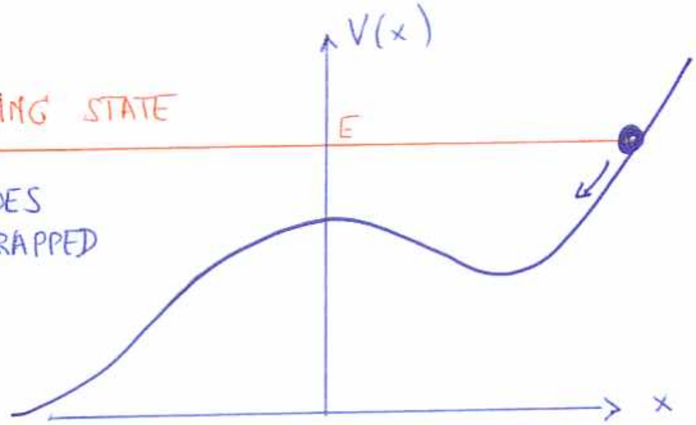
↳ CLASSICAL ↙ ↘ QUANTUM

BOUND STATE
↓
PARTICLE "TRAPPED"



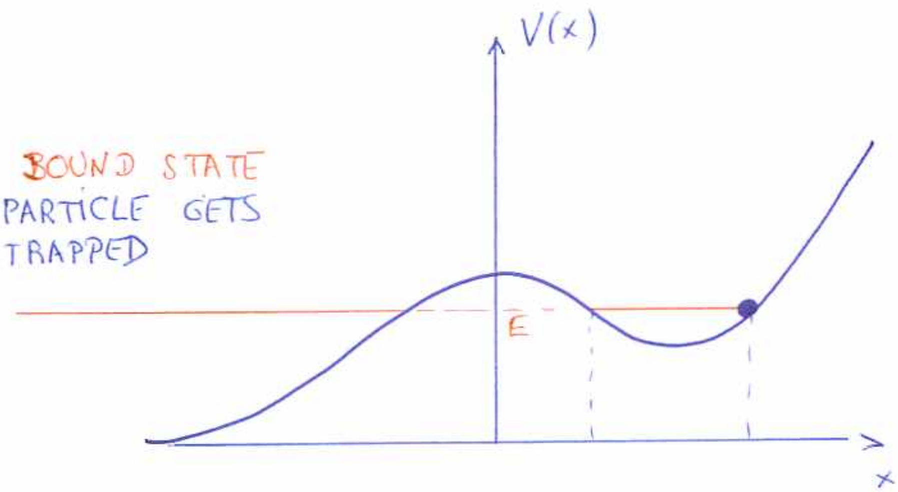
BOUND STATE

SCATTERING STATE
↓
PARTICLE DOES NOT GET TRAPPED



SCATTERING STATE

BOUND STATE
PARTICLE GETS TRAPPED



SCATTERING STATE
↓
PARTICLE CAN 'TUNNEL' THROUGH FINITE BARRIER AND ESCAPE

↳ QUANTUM :

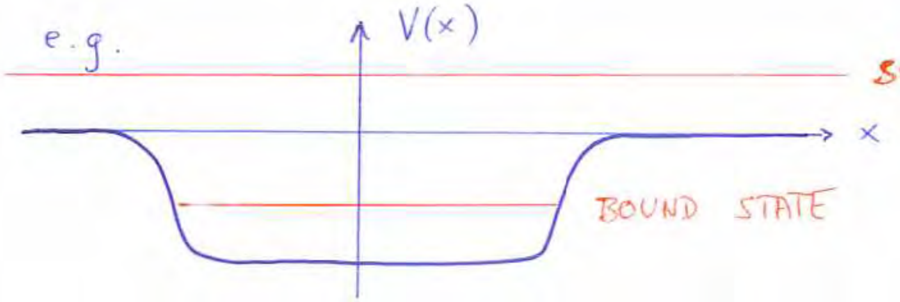
BOUND STATE $\quad E < \left[V(x=+\infty) \;\&\; V(x=-\infty) \right]$

SCATTERING STATE $\quad E > \left[V(x=+\infty) \text{ OR } V(x=-\infty) \right]$

IN PHYSICAL APPLICATIONS :

OFTEN $V(x = \pm\infty) = 0$

e.g.



POTENTIAL FELT BY NUCLEON (PROTON OR NEUTRON) IN THE ATOMIC NUCLEUS.

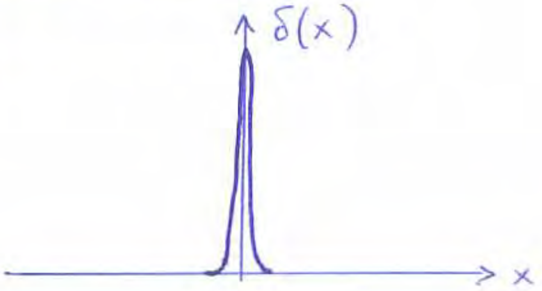
CHARACTERIZED BY

$E < 0$: BOUND STATE $\Rightarrow$ DISCRETE INDEX n

$E > 0$: SCATTERING STATE $\Rightarrow$ CONTINUOUS VARIABLE k

δ - FUNCTION WELL

↳ SPIKE



$$\delta(x) = \begin{cases} 0 & , \; x \neq 0 \\ \infty & , \; x = 0 \end{cases}$$

WITH SURFACE = 1

$$\int_{-\infty}^{+\infty} dx \; \delta(x) = 1.$$

↳ IN MATHEMATICS :

$\delta(x)$ IS NOT A FUNCTION

BUT A 'DISTRIBUTION'

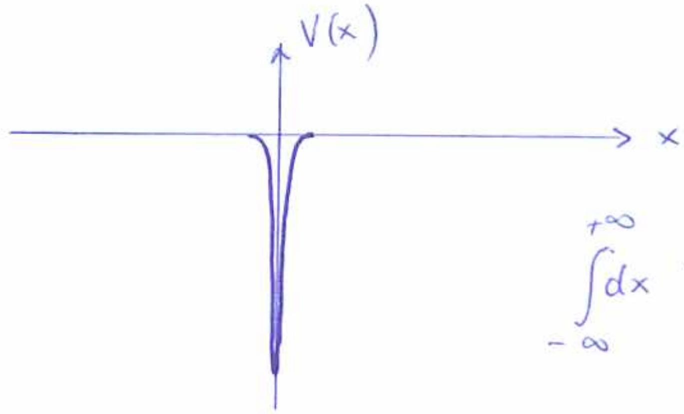
↳ $\delta(x-a)$ SPIKE POSITIONED AT $x = a$

↳ PROPERTIES

$$f(x) \cdot \delta(x-a) = f(a)\, \delta(x-a)$$

$$\int_{-\infty}^{+\infty} dx \; f(x) \, \delta(x-a) = f(a) \int_{-\infty}^{+\infty} dx \; \delta(x-a) = f(a)$$

- $V(x) = -\alpha\,\delta(x)$ WITH $\alpha > 0$



$$\int_{-\infty}^{+\infty} dx\, V(x) = -\alpha.$$

↳ TIME - INDEPENDENT SCHRÖDINGER EQ.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi = E\,\psi$$

SOLUTIONS WITH $E < 0$: BOUND STATES

SOLUTIONS WITH $E > 0$: SCATTERING STATES.

- BOUND STATE SOLUTIONS : $E < 0$

↳ INTRODUCE $E \equiv -\frac{\hbar^2 K^2}{2m} \Rightarrow \boxed{K \equiv \frac{\sqrt{-2mE}}{\hbar}}$ $K > 0$

$$\frac{d^2\psi}{dx^2} + \frac{2m\alpha}{\hbar^2}\,\delta(x)\,\psi = K^2\,\psi$$

↳ FOR $x < 0$: $\frac{d^2\psi}{dx^2} = K^2\psi$

↳ SOLUTION $\psi(x) = A e^{-Kx} + B e^{+Kx}$

$\psi(x = -\infty) = 0 \Rightarrow A = 0$

$\therefore$ $\boxed{\psi(x) = B e^{Kx}, \quad x < 0}$

↳ FOR $x > 0$: $\frac{d^2\psi}{dx^2} = K^2 \psi$

↳ SOLUTION $\psi(x) = F e^{-Kx} + G e^{+Kx}$

$\psi(x = +\infty) = 0 \Rightarrow G = 0$

$$\therefore \boxed{\psi(x) = F e^{-Kx}, \quad x > 0}$$

↳ FOR ANY x : ① ψ SHOULD BE CONTINUOUS

② $\frac{d\psi}{dx}$ IS CONTINUOUS EXCEPT WHERE $V = \infty$

$$\psi(x \to 0^-) = \psi(x \to 0^+)$$

APPROACHING ZERO FROM BELOW

APPROACHING ZERO FROM ABOVE

$\psi(x \to 0^-) = B$

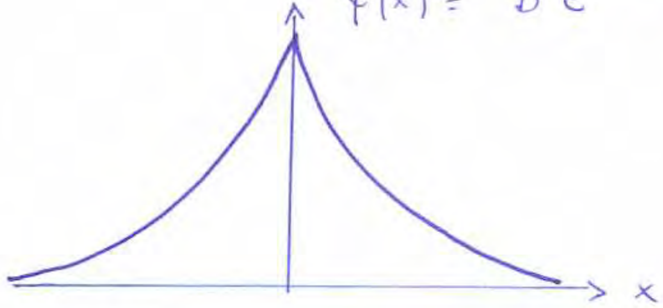
$\psi(x \to 0^+) = F$

$\therefore$ CONTINUITY IMPLIES $\underline{\underline{B = F}}$

$$\boxed{\psi(x) = \begin{cases} B e^{Kx}, & x \leq 0 \\ B e^{-Kx}, & x \geq 0 \end{cases}}$$

2.62

$$\psi(x) = B e^{-K|x|}$$



↳ NORMALIZATION $\displaystyle\int_{-\infty}^{+\infty} dx \quad B^2 e^{-2K|x|} = 1$

$$2B^2 \int_0^{\infty} dx \quad e^{-2Kx} = 1$$

$$\frac{2B^2}{(-2K)} \quad e^{-2Kx} \Big|_0^{\infty} = 1$$

$$B^2 = K \quad \Rightarrow \quad \boxed{B = \sqrt{K}}$$

↳ $\dfrac{d\psi}{dx}$ IS DISCONTINUOUS AT $x = 0$ WHERE $V(x) = -\infty$

INTEGRATE SCHRODINGER EQ. FROM $-\varepsilon$ TO $+\varepsilon$ $(\varepsilon > 0)$

AND LET $\varepsilon \to 0$

$$-\frac{\hbar^2}{2m} \int_{-\varepsilon}^{\varepsilon} dx \; \frac{d^2\psi}{dx^2} + \int_{-\varepsilon}^{+\varepsilon} dx \; V(x)\, \psi(x) = E \int_{-\varepsilon}^{+\varepsilon} dx \; \psi(x)$$

$$-\frac{\hbar^2}{2m} \left(\frac{d\psi}{dx}\Big|_{x=+\varepsilon} - \frac{d\psi}{dx}\Big|_{x=-\varepsilon} \right) + \int_{-\varepsilon}^{+\varepsilon} dx \; V(x)\, \psi(x)$$

$$= E \, \psi(0) \cdot 2\varepsilon$$

$$\xrightarrow[\varepsilon \to 0]{} 0$$

JUMP (DISCONTINUITY) IN DERIVATIVE

$$\Delta\left(\frac{d\psi}{dx}\right)_{x=0} \equiv \left.\frac{d\psi}{dx}\right|_{x=+\varepsilon} - \left.\frac{d\psi}{dx}\right|_{x=-\varepsilon} \quad (\varepsilon \to 0)$$

$$-\frac{\hbar^2}{2m}\,\Delta\left(\frac{d\psi}{dx}\right)_{x=0} = -\int_{-\varepsilon}^{+\varepsilon} dx\; V(x)\,\psi(x)$$

CASE 1 : $V(x)$ IS FINITE AT $x=0$

RHS $\displaystyle\int_{-\varepsilon}^{+\varepsilon} dx\; V(x)\,\psi(x) = 2\varepsilon\, V(0)\,\psi(0) \xrightarrow[\varepsilon\to 0]{} 0$

$\Downarrow$

$$\Delta\left(\frac{d\psi}{dx}\right)_{x=0} = 0 \quad : \quad \frac{d\psi}{dx} \text{ IS CONTINUOUS}$$

CASE 2 : $V(x)$ IS INFINITE AT $x=0$

e.g. $V(x) = -\alpha\,\delta(x)$

RHS $\displaystyle\int_{-\varepsilon}^{+\varepsilon} dx\; V(x)\,\psi(x) = -\alpha\int_{-\varepsilon}^{+\varepsilon} dx\;\delta(x)\,\psi(x)$

$\Downarrow$ $\qquad = -\alpha\,\psi(0)$

$$\boxed{\Delta\left(\frac{d\psi}{dx}\right)_{x=0} = -\frac{2m\alpha}{\hbar^2}\,\psi(0)}$$

↳ APPLY TO OUR SOLUTION

$$\psi(x) = \begin{cases} \sqrt{K}\, e^{Kx}, & x \leqslant 0 \\ \sqrt{K}\, e^{-Kx}, & x \geqslant 0 \end{cases}$$

$$\left.\frac{d\psi}{dx}\right|_{x=-\varepsilon} = K\sqrt{K}\, e^{-K\varepsilon} \xrightarrow[\varepsilon \to 0]{} K^{3/2}$$

$$\left.\frac{d\psi}{dx}\right|_{x=+\varepsilon} = -K\sqrt{K}\, e^{-K\varepsilon} \xrightarrow[\varepsilon \to 0]{} -K^{3/2}$$

$$\Delta\left(\frac{d\psi}{dx}\right)_{x=0} = -2K^{3/2}$$

$$\therefore \quad -2K^{3/2} = -\frac{2m\alpha}{\hbar^2} \cdot \underbrace{\sqrt{K}}_{\psi(0)}$$

$$\Downarrow$$

$$\boxed{K = \frac{m\alpha}{\hbar^2}}$$

BOUND STATE ENERGY

$$\boxed{E = -\frac{\hbar^2 K^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}}$$

δ - FUNCTION POTENTIAL HAS 1 BOUND STATE

$$\psi(x) = \sqrt{K}\, e^{-K|x|}$$

- **SCATTERING SOLUTIONS : $E > 0$**

↳ INTRODUCE $E = \dfrac{\hbar^2 k^2}{2m} \Rightarrow \boxed{k \equiv \dfrac{\sqrt{2mE}}{\hbar}}$

$$\frac{d^2\psi}{dx^2} + \frac{2m\alpha}{\hbar^2}\,\delta(x)\,\psi = -k^2\psi$$

↳ FOR $x < 0$: $\dfrac{d^2\psi}{dx^2} = -k^2\psi$

↳ SOLUTION $\psi(x) = A e^{ikx} + B e^{-ikx}$

NONE OF THE TERMS BLOWS UP FOR $x \to -\infty$

↳ FOR $x > 0$: $\psi(x) = F e^{ikx} + G e^{-ikx}$

↳ CONTINUITY AT $x = 0$

$$\psi(0^-) = A + B = \psi(0^+) = F + G$$

↳ JUMP IN DERIVATIVE

$$\left.\frac{d\psi}{dx}\right|_{x=0^-} = ik\,(A - B)$$

$$\left.\frac{d\psi}{dx}\right|_{x=0^+} = ik\,(F - G)$$

$$\Delta \left(\frac{d\psi}{dx} \right)_{x=0} = - \frac{2m\alpha}{\hbar^2} \psi(0)$$

$$\Downarrow$$

$$ik \left(F - G - A + B \right) = - \frac{2m\alpha}{\hbar^2} \left(A + B \right)$$

↳ $\underline{\underline{F - G = A(1 + 2i\beta) - B(1 - 2i\beta)}}$

$$\boxed{\beta \equiv \frac{m\alpha}{\hbar^2 k}}$$

↳ PROBLEM : 4 UNKNOWNS A, B, F, G

ONLY 2 EQUATIONS.

(FOR BOUND STATES 2 UNKNOWNS WERE ELIMINATED FROM ASYMPTOTIC CONDITIONS)

↳

$e^{ikx} \cdot e^{-\frac{i}{\hbar}Et}$: WAVE TRAVELING ⟶

$e^{-ikx} \cdot e^{-\frac{i}{\hbar}Et}$: WAVE TRAVELING ⟵

IN AN EXPERIMENT A WAVE IS TRAVELING FROM ONE SIDE TO THE OTHER.

SAY WE CONSIDER WAVE TRAVELING ⟶ STARTING FROM $x = -\infty$

$\therefore$ $\underbrace{G = 0}_{\text{NO WAVE IS COMING IN FROM } x = +\infty}$ (BOUNDARY CONDITIONS)

$$\begin{cases} F = A + B \\ F = A(1 + 2i\beta) - B(1 - 2i\beta). \end{cases}$$

$$\hookrightarrow \quad A + B = A(1 + 2i\beta) - B(1 - 2i\beta)$$

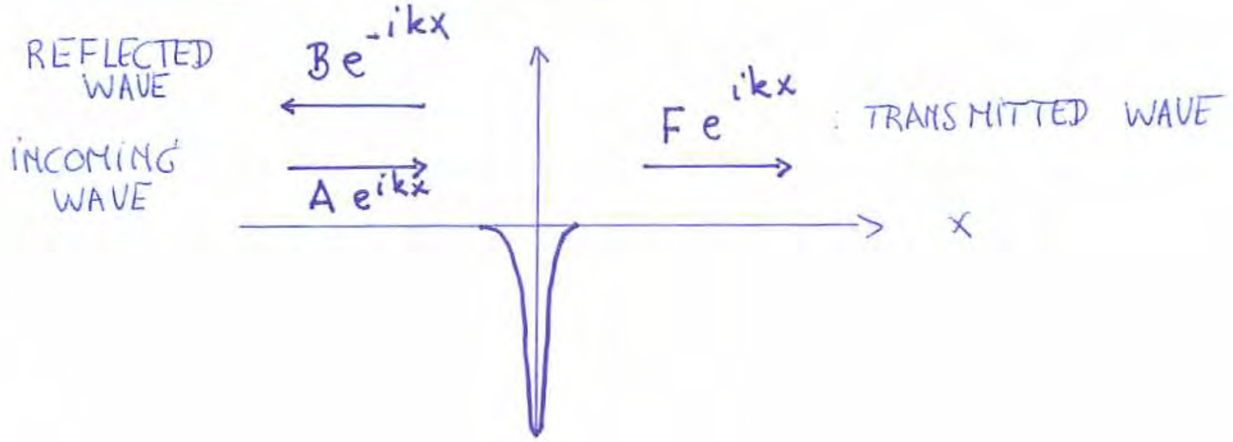
$$\Downarrow$$

$$(1 - i\beta)\, 2B = 2i\beta\, A$$

$$\boxed{B = \frac{i\beta}{1 - i\beta} A}$$

$$\hookrightarrow \quad F = A + B = \frac{1 - i\beta + i\beta}{1 - i\beta} A$$

$$\boxed{F = \frac{1}{1 - i\beta} A}$$



↳ PROBABILITY OF REFLECTION $|\Psi|^2$

$$\boxed{R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}}$$

↳ PROBABILITY OF TRANSMISSION

$$\boxed{T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}}$$

↳ $\underline{\underline{R + T = 1}}$

↳ IN TERMS OF α :

$$R = \frac{1}{1+1/\beta^2} = \frac{1}{1+\frac{\hbar^4 k^2}{m^2\alpha^2}} = \frac{1}{1+\frac{2\hbar^2 E}{m\alpha^2}}$$

$$T = \frac{1}{1+\beta^2} = \frac{1}{1+\frac{m\alpha^2}{2\hbar^2 E}}$$

$E \gg$ (VERY LARGE) $\quad R \to 0 \;,\; T \to 1 \quad$ PERFECT TRANSMISSION

$E \ll$ (VERY LOW) $\quad R \to 1 \;,\; T \to 0 \quad$ PERFECT REFLECTION

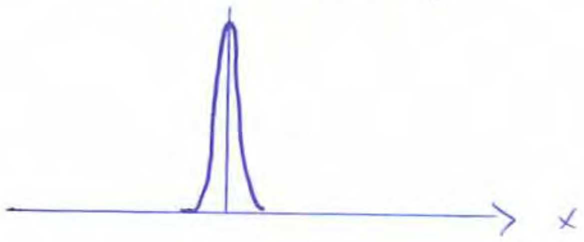
↳ NORMALIZATION

SCATTERING STATES OF DEFINITE k (PLANE WAVES) ARE NOT NORMALIZABLE.

↳ PHYSICAL PARTICLES ARE DESCRIBED BY LINEAR SUPERPOSITIONS OF PLANE WAVES ⇒ WAVE PACKETS

- **δ - FUNCTION BARRIER**

$$V(x) = +\alpha\,\delta(x)$$



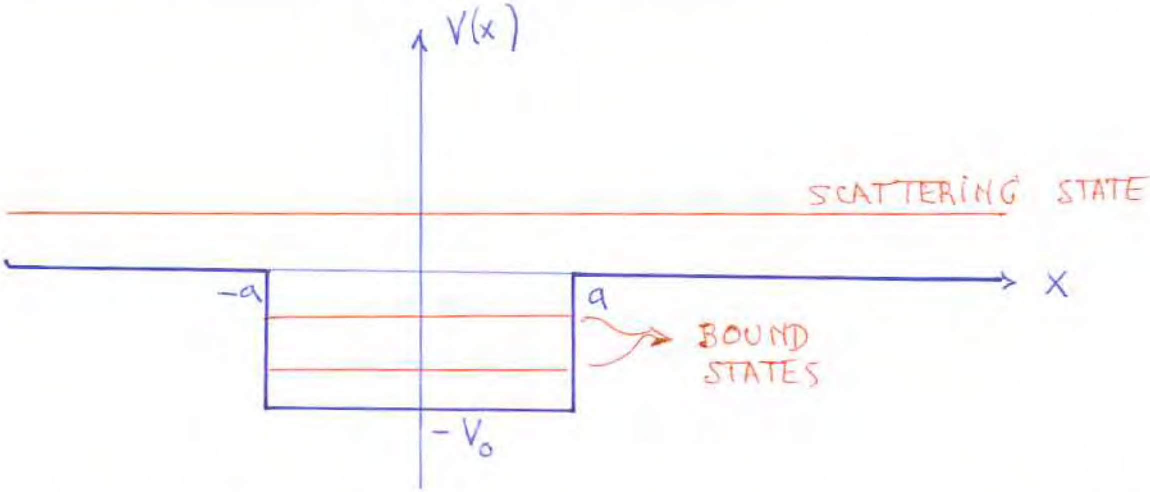
PUT $\alpha \to -\alpha$ IN PREVIOUS

↳ NO BOUND STATE

↳ EXACTLY SAME SCATTERING STATES.

R & T ONLY DEPEND ON α^2 !

⇒ FINITE SQUARE WELL

• 

$$V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & |x| > a \end{cases} \qquad (V_0 > 0)$$

↳ POTENTIAL ALLOWS FOR BOTH ↗ BOUND STATES $(E < 0)$
↘ SCATTERING STATES $(E > 0)$

• **BOUND STATES** $(E<0)$

↳ $x < -a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\downarrow \qquad E \equiv -\frac{\hbar^2 \kappa^2}{2m}, \quad \boxed{\kappa = \frac{\sqrt{-2mE}}{\hbar}}$$

$$\psi(x) = \cancel{Ae^{-\kappa x}} + Be^{\kappa x}$$

$$\psi(x \to -\infty) = 0$$

↳ $x > a$ ANALOGOUSLY

$$\psi(x) = Fe^{-\kappa x} + \cancel{Ge^{\kappa x}} \qquad \psi(x \to +\infty) = 0$$

↳ $-a < x < a$: $V(x) = -V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

$$\Downarrow \qquad \boxed{l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}}$$

$$\frac{d^2\psi}{dx^2} = -l^2\psi$$

$E < 0$

BUT $V_0 + E > 0$

$(E > -V_0)$

$$\Downarrow$$

$$\psi(x) = C\sin(lx) + D\cos(lx)$$

↳ BOUNDARY CONDITIONS

ψ & $\dfrac{d\psi}{dx}$ CONTINUOUS AT BOUNDARIES $x = \pm a$

SYMMETRIC POTENTIAL : SOLUTIONS EITHER EVEN OR ODD

$$\psi(-x) = \pm\psi(x)$$

- EVEN SOLUTIONS $\psi(-x) = \psi(x)$

BOUNDARY CONDITION AT $x = +a$

$$\psi(x) = \begin{cases} D\cos(lx), & 0 < x < a \\ Fe^{-Kx}, & x > a \end{cases}$$

$\Rightarrow$ CONTINUITY OF ψ AT $x = a$

$$\Downarrow$$

$$Fe^{-Ka} = D\cos(la) \qquad (*)$$

$\Rightarrow$ CONTINUITY OF $\dfrac{d\psi}{dx}$ AT $x = a$.

$$\Downarrow$$

$$-FKe^{-Kx} = -D\ell \sin(\ell a) \qquad (**)$$

$$\frac{(**)}{(*)} \Rightarrow \boxed{+\ell \tan(\ell a) = K}$$

$\hookrightarrow$ TRANSCENDENTAL EQ.
TO BE SOLVED TO DETERMINE E

DEFINE
$$z \equiv \ell a = \frac{a}{\hbar}\sqrt{2m(E+V_0)}$$
$$z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}$$

$$z_0^2 - z^2 = \frac{a^2}{\hbar^2}(-2mE) = a^2K^2$$

$$aK = \sqrt{z_0^2 - z^2}$$

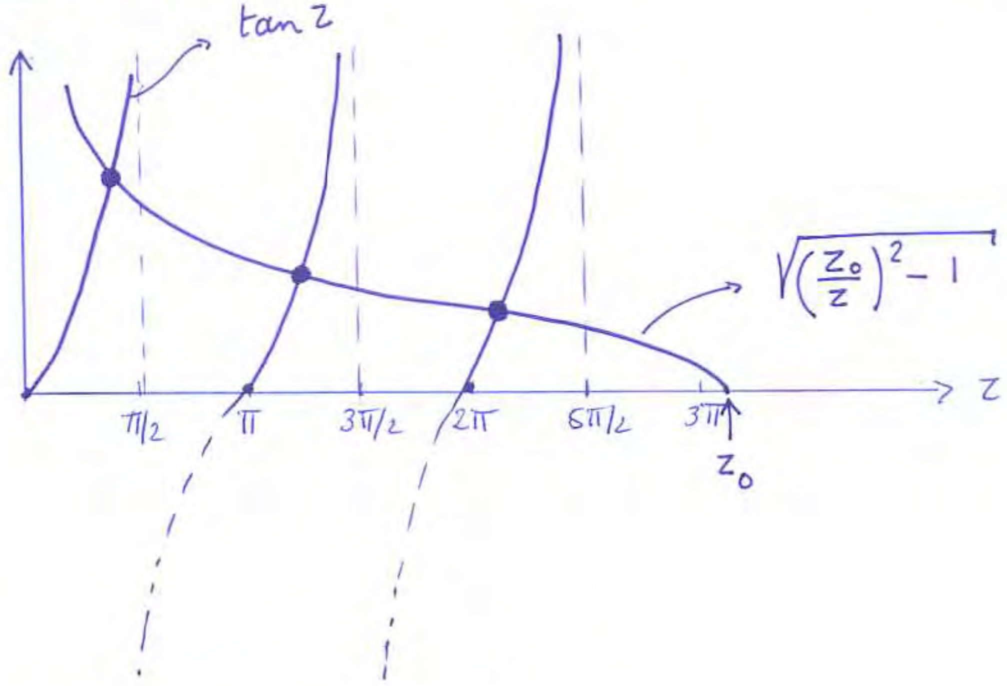
$$\therefore \quad a\ell \tan(\ell a) = aK$$

$$\Updownarrow$$

$$z \tan z = \sqrt{z_0^2 - z^2}$$

$$\boxed{\tan z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}}$$

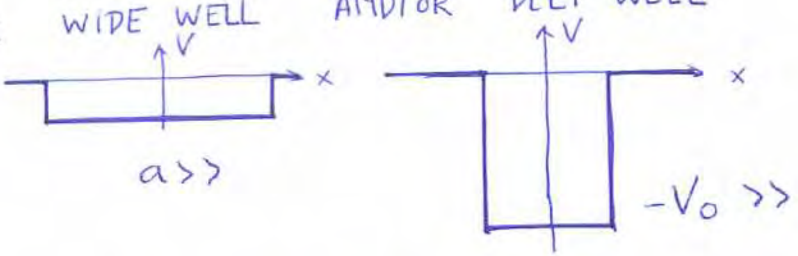
$\Rightarrow$ GRAPHICAL SOLUTION



INTERSECTIONS ARE SOLUTIONS.

$\Rightarrow$ SPECIAL LIMITING CASES.

- $z_0 \gg$: WIDE WELL AND/OR DEEP WELL



SOLUTIONS $\quad z \simeq z_n = n\frac{\pi}{2} \qquad (n \text{ ODD})$

$$\underbrace{E_n + V_0}_{} \simeq \frac{\hbar^2}{2m}\,\frac{1}{a^2}\cdot\left(n\frac{\pi}{2}\right)^2$$

ENERGY ABOVE BOTTOM OF WELL

$$\boxed{E_n = -V_0 + \frac{\hbar^2\,\pi^2}{2m\,(2a)^2}\cdot n^2}$$

$(n \text{ ODD})$

FOR V_0 FINITE $\Rightarrow$ ONLY FINITE # BOUND STATES

FOR $V_0 \to +\infty \Rightarrow \infty$ # BOUND STATES

THIS IS INFINITE SQUARE WELL
OF WIDTH $(2a)$ FOR n ODD

$$E_1, E_3, E_5, \ldots$$

(EVEN n CORRESPOND WITH
ODD WAVE FUNCTIONS $\psi(-x) = -\psi(x)$)

- $Z_0 \ll$ NARROW WELL AND/OR SHALLOW WELL
  $a \ll$ $\qquad$ $-V_0 \ll$

$Z_0 \downarrow$ LESS & LESS BOUND STATES.

FOR $Z_0 < \frac{\pi}{2} \Rightarrow$ ONLY 1 BOUND STATE REMAINS

$\therefore$ THERE IS ALWAYS 1 BOUND STATE
EVEN FOR SUPERWEAK POTENTIAL
OR VERY NARROW WELL

- **ODD SOLUTIONS** $\psi(-x) = -\psi(x)$

BOUNDARY CONDITION AT $x = +a$

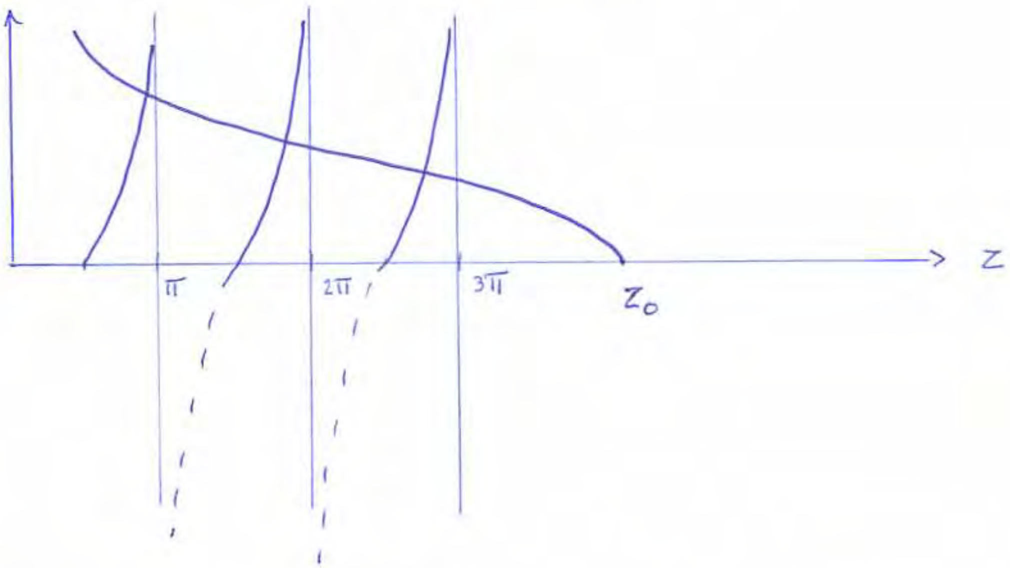
$$\psi(x) = \begin{cases} C \sin lx & , \quad 0 < x < a \\ F e^{-Kx} & , \quad x > a \end{cases}$$

$$\Rightarrow \quad F e^{-Ka} = C \sin lx$$

$$\Rightarrow \quad -F K e^{-Ka} = C l \cos lx$$

$$\therefore \quad \boxed{l \cot(la) = -K}$$

$$\boxed{-\cot z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}}$$



SOLUTIONS $\quad z_n \simeq n\pi \qquad n = 1, 2, 3 \dots$

$$= (2n)\frac{\pi}{2}$$

NO BOUND STATE IF $\boxed{z_0 < \frac{\pi}{2} \iff V_0 < \frac{\pi^2 \hbar^2}{8ma^2}}$

- **SCATTERING STATES** $(E > 0)$

* ASSUMING: INCIDENT WAVE MOVING FROM $x=-\infty$ TO RIGHT

↳ $x < -a$ $\quad V(x) = 0$

$$\boxed{k \equiv \frac{\sqrt{2mE}}{\hbar}}$$

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

↳ INCIDENT AMPLITUDE

↳ $-a < x < a$ $\quad V(x) = -V_0$

$$\psi(x) = C \sin(\ell x) + D \cos(\ell x)$$

$$\boxed{\ell = \frac{1}{\hbar}\sqrt{2m(E+V_0)}}$$ AS BEFORE

↳ $x > a$ $\quad V(x) = 0$

$$\psi(x) = F e^{ikx}$$

(TERM $\sim e^{-ikx}$ ABSENT FOR INCIDENT WAVE MOVING FROM $x = -\infty$ TO RIGHT)

A: INCIDENT AMPLITUDE

B: REFLECTED AMPLITUDE

F: TRANSMITTED AMPLITUDE

↳ BOUNDARY CONDITIONS

CONTINUITY OF ψ AND $\frac{d\psi}{dx}$ AT $x=-a$ AND $x=+a$

(1) • $\psi(-a)$: $\quad Ae^{-ika} + Be^{+ika} = -C\sin(la) + D\cos(la)$

(2) • $\frac{d\psi}{dx}(-a)$: $\quad ik\left[Ae^{-ika} - Be^{+ika}\right] = l\left[+C\cos(la) + D\sin(la)\right]$

(3) • $\psi(+a)$: $\quad Fe^{ika} = C\sin(la) + D\cos(la)$

(4) • $\frac{d\psi}{dx}(+a)$: $\quad ikFe^{ika} = l\left[C\cos(la) - D\sin(la)\right]$

(3) $\cdot\, l\sin(la)$ + (4) $\cos(la)$

↳ $lC = Fe^{ika}\left[l\sin(la) + ik\cos(la)\right]$

$$\boxed{C = Fe^{ika}\left[\sin(la) + i\frac{k}{l}\cos(la)\right]}$$

(3) $l\cos(la)$ − (4) $\sin(la)$

↳ $$\boxed{D = Fe^{ika}\left[\cos(la) - i\frac{k}{l}\sin(la)\right]}$$

PLUG C, D INTO (1) & (2)

$$(*) \quad A e^{-ika} + B e^{ika}$$

$$= F e^{ika} \left\{ - \sin^2(la) - i \frac{k}{l} \sin(la) \cos(la) + \cos^2(la) - i \frac{k}{l} \sin(la) \cos(la) \right\}$$

$$= F e^{ika} \left\{ \cos(2la) - i \frac{k}{l} \sin(2la) \right\}$$

$$(**) \quad ik \left[A e^{-ika} - B e^{+ika} \right]$$

$$= F e^{ika} l \left\{ + \sin(la) \cos(la) + i \frac{k}{l} \cos^2(la) + \sin(la) \cos(la) - i \frac{k}{l} \sin^2(la) \right\}$$

$$= F e^{ika} l \left\{ + \sin(2la) + i \frac{k}{l} \cos(2la) \right\}$$

$\hookrightarrow \quad ik(*) - (**)$

$$B e^{ika} \cdot 2ik = F e^{ika} \left\{ \cancel{ik \cos(2la)} + \frac{k^2}{l} \sin(2la) - l \sin(2la) - \cancel{ik \cos(2la)} \right\}$$

$$= F e^{ika} \frac{k^2 - l^2}{l} \sin(2la)$$

$$\boxed{B = i \frac{\sin(2la)}{2kl} (l^2 - k^2) F}$$

$\hookrightarrow \quad ik\,(*) + (**)$

$$A e^{-ika} \cdot 2ik = F e^{ika} \Big\{ ik \cos(2la) + \frac{k^2}{l} \sin(2la)$$

$$+ \, l \sin(2la) + ik \cos(2la) \Big\}$$

$$= F e^{ika} \Big\{ 2ik \cos(2la) + \frac{k^2+l^2}{l} \sin(2la) \Big\}$$

$$\Downarrow$$

$$\boxed{F = A \cdot \frac{e^{-2ika}}{\cos(2la) - i \dfrac{(k^2+l^2)}{2kl} \sin(2la)}}$$

$\hookrightarrow$ R & T

$$R = \frac{|B|^2}{|A|^2} \quad , \quad T = \frac{|F|^2}{|A|^2} \quad , \quad R + T = 1$$

$$T^{-1} = \cos^2(2la) + \frac{(k^2+l^2)^2}{4k^2l^2} \sin^2(2la)$$

$$= 1 + \frac{(k^2-l^2)^2}{4k^2l^2} \sin^2(2la).$$

$$\ell^2 - k^2 = \frac{1}{\hbar^2} 2mV_0$$

$$\Downarrow$$

$$\boxed{T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}$$

$T \leqslant 1$

$\Rightarrow$ REACHES 1 FOR ZEROES OF $\sin^2(\ )$

$$\Downarrow$$

$$T = 1 \Leftrightarrow \boxed{\frac{2a}{\hbar}\sqrt{2m(E_n+V_0)} = n\pi} \qquad n \text{ INTEGER}$$

↳ FOR THESE ENERGY : WELL BECOMES PERFECTLY TRANSPARANT

$$\Updownarrow$$

$$2\ell a = n\pi$$

$e^{i\ell a} \Longleftrightarrow$ WAVE WITH WAVELENGTH $\lambda = \frac{2\pi}{\ell}$

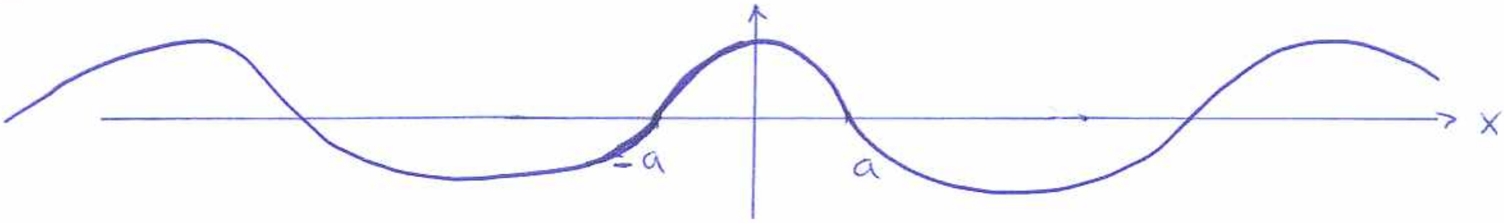
PERFECT TRANSPARANT FOR $2 \cdot \frac{2\pi}{\lambda} \cdot a = n\pi$

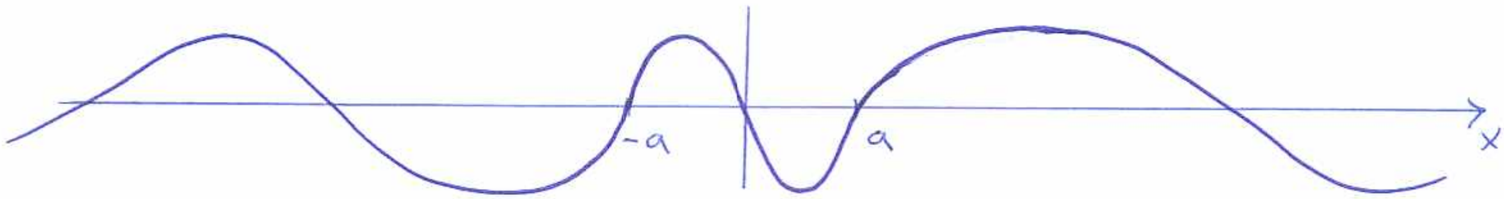
$$\Updownarrow$$

$$T = 1 \Leftrightarrow \boxed{(2a) = n\frac{\lambda}{2}}$$

∇ CONDITION FOR PERFECT TRANSPARANCY ($T = 1$)

$\circ$ WHEN WIDTH OF WELL FITS AN INTEGER TIMES $\lambda/2$



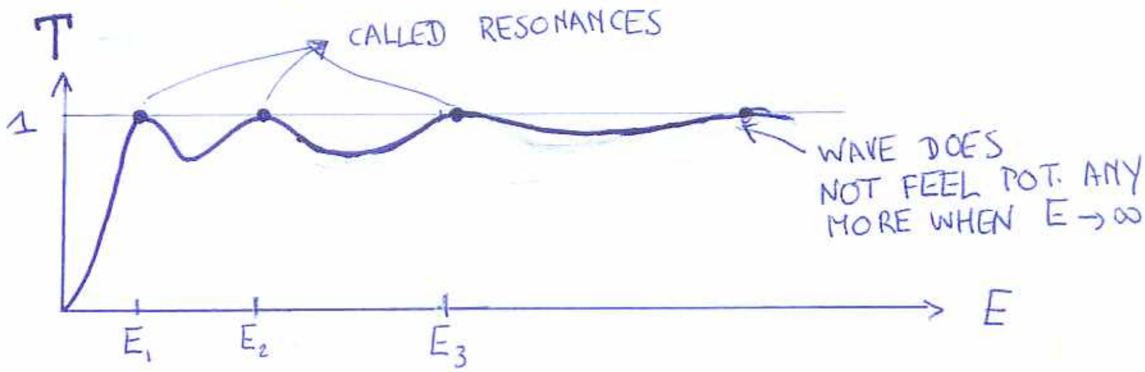


ENERGIES FOR PERFECT TRANSMISSION

$$\underbrace{E_n = -V_0 + n^2 \frac{\pi^2 \hbar^2}{2m\,(2a)^2}}, \quad n \text{ INTEGER } n = 1, 2, \ldots$$

(cf. ENERGIES OF INFINITE SQUARE WELL)

$\Rightarrow$ PLOT



T HAS MINIMA FOR $\sin(2a\ell) = 1$

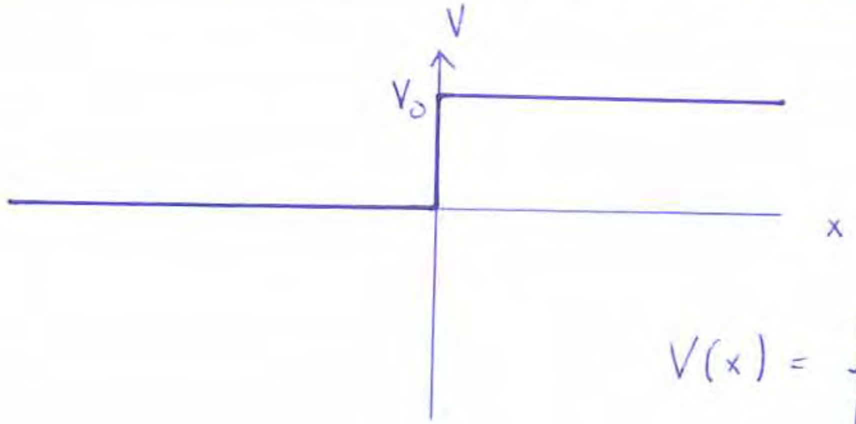
$$\Updownarrow$$

$$2a \cdot \frac{2\pi}{\lambda} = n' \frac{\pi}{2} \qquad (n' = 1, 3, 5, \ldots)$$

$$\downarrow$$

$$\boxed{2a = n' \cdot \frac{\lambda}{4}} \quad (n' = 1, 3, 5 \ldots)$$

⇒ FURTHER EXAMPLE : STEP POTENTIAL



$$V(x) = \begin{cases} 0, & x \leq 0 \\ V_0, & x > 0 \end{cases}$$

- $E < V_0$

↳ $x < 0$. SOLUTION $\psi(x) = A e^{ikx} + B e^{-ikx}$

$$\boxed{k \equiv \frac{\sqrt{2mE}}{\hbar}}$$

↳ $x > 0$ $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0 \psi = E \psi$

↓

$$\frac{d^2\psi}{dx^2} = + \frac{2m}{\hbar^2}(V_0 - E)\psi$$

↓ $V_0 > E$

$$\boxed{K \equiv \frac{\sqrt{2m(V_0 - E)}}{\hbar}}$$

$\psi(x) = C e^{-Kx}$

↳ BOUNDARY CONDITION $x = 0$

ψ CONTINUOUS : $A + B = C$

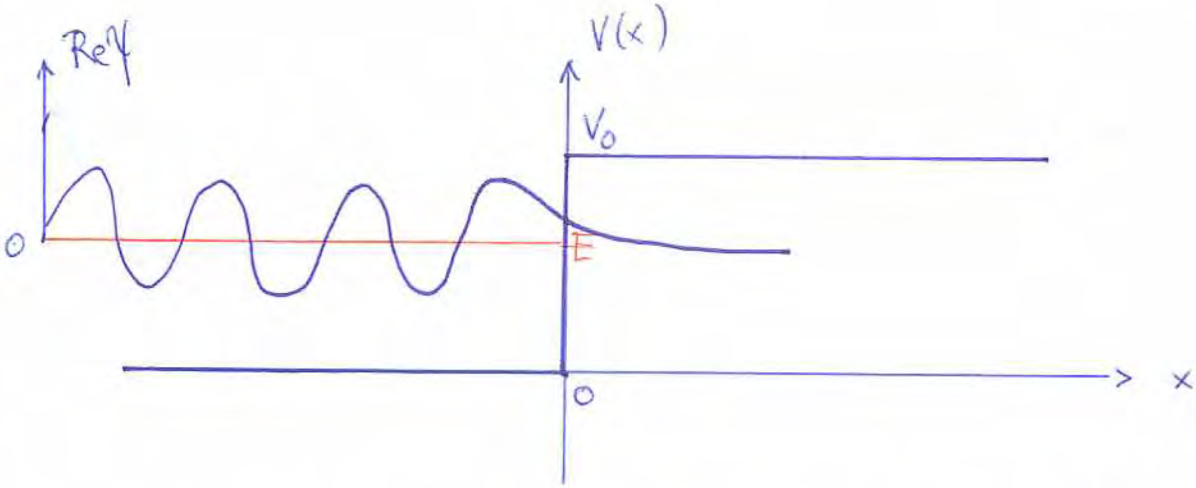
$\frac{d\psi}{dx}$ CONTINUOUS : $ik(A - B) = -CK$

$$\Downarrow$$

$$\frac{A-B}{A+B} = +\frac{i}{k}K$$

$$\boxed{B = A \cdot \frac{1 - \frac{i}{k}K}{1 + \frac{i}{k}K}}$$

REFLECTION COEFF. $R = \frac{|B|^2}{|A|^2} = 1$ TOTAL REFLECTION



$$E = 0 \Rightarrow k = 0 \Rightarrow B = -A$$

$$E = V_0 \Rightarrow K = 0 \Rightarrow B = +A$$

- $E > V_0$

↳ $\underline{x<0}$: $\psi(x) = A e^{ikx} + B e^{-ikx}$

↳ $\underline{x>0}$: $\psi(x) = D e^{i\ell x}$ FOR INCOMING WAVE FROM $x = -\infty$ ⟶

$$\boxed{\ell \equiv \frac{1}{\hbar}\sqrt{2m(E-V_0)}}$$

↳ BOUNDARY CONDITION $x = 0$

ψ : $A + B = D$

$\frac{d\psi}{dx}$: $ik(A-B) = i\ell D$

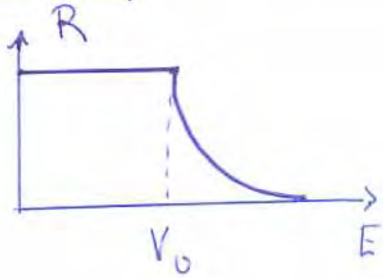
$$\Downarrow$$

$$k(A-B) = \ell(A+B)$$

$$\boxed{B = A \cdot \frac{(k-\ell)}{(k+\ell)}} \implies D = A + B = A\frac{2k}{(k+\ell)}$$

↳ REFLECTION COEFF. $\boxed{R = \frac{|B|^2}{|A|^2} = \frac{(k-\ell)^2}{(k+\ell)^2}}$

$$R = \frac{\left(\sqrt{E} - \sqrt{E-V_0}\right)^2}{\left(\sqrt{E} + \sqrt{E-V_0}\right)^2} = \frac{\left(\sqrt{E} - \sqrt{E-V_0}\right)^4}{V_0^{\,2}}$$



↳ TRANSMISSION COEFF. $T = 1 - R$

$$T = 1 - \frac{(k-\ell)^2}{(k+\ell)^2}$$

$$= \frac{(k^2 + 2k\ell + \ell^2 - k^2 + 2k\ell - \ell^2}{(k+\ell)^2}$$

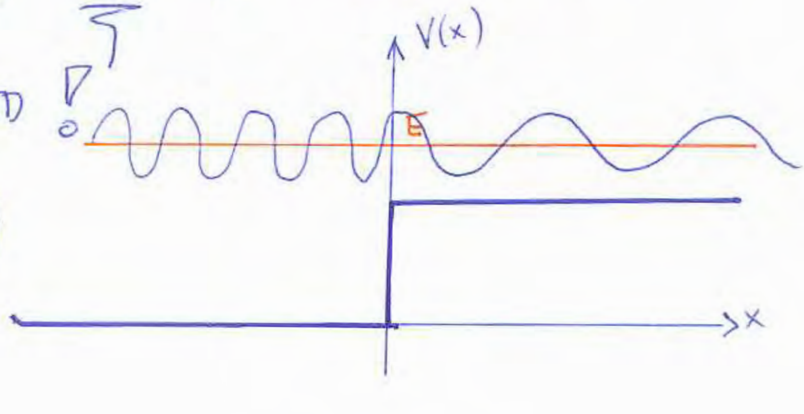
$$= \frac{4k\ell}{(k+\ell)^2}$$

$$= \frac{\ell}{k} \cdot \underbrace{\left(\frac{4k^2}{(k+\ell)^2}\right)}_{\frac{|D|^2}{|A|^2}}$$

$$\boxed{T = \frac{\ell}{k} \cdot \frac{|D|^2}{|A|^2}} = \frac{\sqrt{E - V_0}}{\sqrt{E}} \cdot \frac{|D|^2}{|A|^2}$$

↳ NOTE T COEFF. IS ONLY OBTAINED AS RATIO OF $|\ |^2$ IF BOTH WAVES TRAVEL AT SAME SPEED !

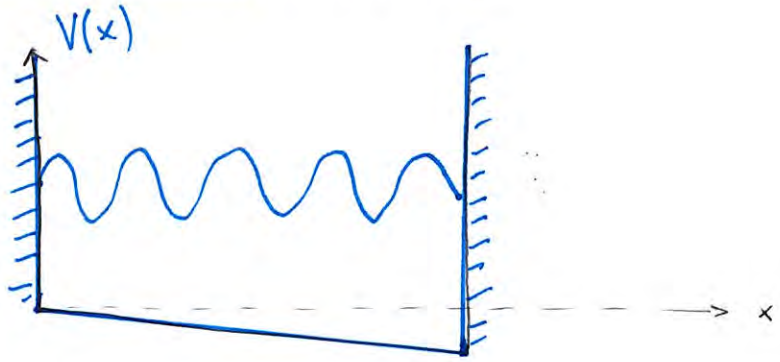
(SAME FOR R)



⇒ 2.7 WKB APPROXIMATION

- SEMI-CLASSICAL METHOD BY WENTZEL, KRAMERS, BRILLOUIN (WKB) TO TIME-INDEPENDENT SCHRÖDINGER EQ. WHEN V(x) VARIES VERY SLOWLY IN COMPARISON WITH λ (WAVELENGTH)

e. g.



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{1}{\hbar^2}\,2m\,(E - V(x))\,\psi$$

$$\boxed{k(x) \equiv \frac{1}{\hbar}\sqrt{2m\big(E - V(x)\big)}}$$ FOR $E > V(x)$

$$\psi(x) = A(x)\,e^{i\phi(x)}$$

$A(x)$ & $\phi(x)$ ARE REAL FUNCTIONS

$$\psi' = \left(A' + i\phi' A \right) e^{i\phi}$$

$$\psi'' = \left(A'' + 2i\,\phi' A' + i A \phi'' - A(\phi')^2 \right) e^{i\phi}$$

SCHRÖDINGER EQ.

$$A'' + 2i\,A'\phi' + i A\phi'' - A(\phi')^2 = -k^2(x)\,A$$

REAL PART : $A'' - A(\phi')^2 = -k^2(x)\,A$

IM. PART $2A'\phi' + A\phi'' = 0$

$$\Downarrow$$

$$\frac{d}{dx}\left(A^2 \phi' \right) = 0$$

CONSTANT

$$\boxed{A(x) = \frac{C}{\sqrt{\phi'(x)}}}$$

APPROXIMATION FOR REAL PART

VARIATION OF $A(x)$ SLOW COMPARED TO λ

$$\Downarrow$$

$$A''(x) \simeq 0$$

$$\phi'^2 = k^2(x)$$

$$\boxed{\frac{d\phi}{dx} = \pm k(x)}$$

$$\phi(x) = \pm \int dx \; k(x)$$

$$\boxed{\psi(x) \simeq \frac{C}{\sqrt{k(x)}} \exp\left\{\pm i \int dx \; k(x)\right\}}$$

$$|\psi(x)|^2 \simeq \frac{|C|^2}{k(x)}$$

PROBABILITY INVERSELY PROP. TO MOMENTUM (OR VELOCITY) AT A POINT x $\rightsquigarrow$ cfr. CLASSICAL RESULT

• EXAMPLE : INFINITE WELL WITH SLOWLY VARYING BOTTOM

$$\psi(x) \simeq \frac{1}{\sqrt{k(x)}} \left\{ C_+ e^{i\phi(x)} + C_- e^{-i\phi(x)} \right\}$$

OR EQUIVALENTLY

$$\simeq \frac{1}{\sqrt{k(x)}} \left\{ C_1 \sin \phi(x) + C_2 \cos \phi(x) \right\}$$

BOUNDARY CONDITIONS :

$$\psi(0) = 0$$

$$\psi(a) = 0$$

$$\phi(x) = \int_0^x dx \quad k(x)$$

$\hookrightarrow$ SINCE $\phi(0) = 0 \implies \psi(0) = 0 \iff C_2 = 0$

$\hookrightarrow$ $\psi(a) = 0 \implies \phi(a) = n\pi \qquad (n = 1, 2, \ldots)$

$$\boxed{\int_0^a dx \quad k(x) = n\pi}$$

$\hookrightarrow$ SPECIAL CASE: FOR $V(x) = 0$

$$\downarrow$$

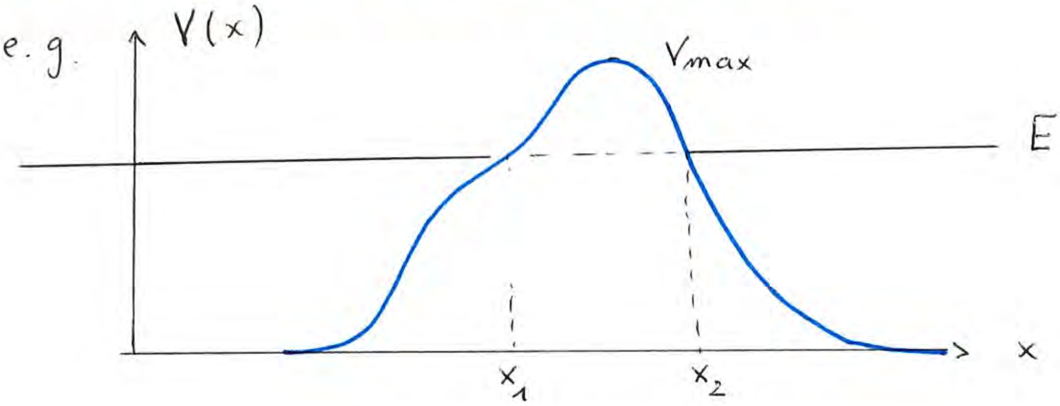
$$k(x) = k = \frac{1}{\hbar}\sqrt{2mE} \qquad \text{CONSTANT}$$

$$k \cdot a = n\pi$$

$$k = \frac{n\pi}{a} \qquad \text{(RESULT OF BEFORE)}$$

TUNNELING THROUGH A BARRIER

e.g.



FOR $E < V_{max}$

$$\frac{d^2\psi}{dx^2} = \frac{1}{\hbar^2}\, 2m\left[V(x) - E\right]\psi$$

FOR $x_1 < x < x_2$: CLASICALLY FORBIDDEN REGION

$$K(x) \equiv \frac{1}{\hbar}\sqrt{2m\left(V(x)-E\right)}$$

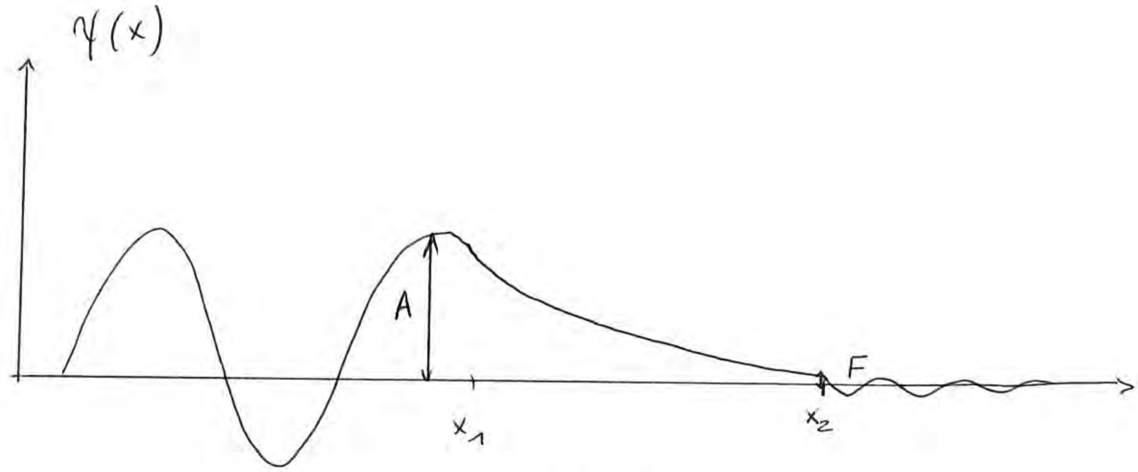
$$\psi(x) \simeq \frac{C_1}{\sqrt{K(x)}}\exp\left\{\int dx\; K(x)\right\}$$

$$+ \frac{C_2}{\sqrt{K(x)}}\exp\left\{-\int dx\; K(x)\right\}$$

$\downarrow$ FOR $x_2 - x_1 >> \lambda$ VERY THICK WELL

$C_1 \simeq 0$ (OTHERWISE EXP. INCREASING)

$$\simeq \frac{C_2}{\sqrt{K(x)}}\exp\left\{-\int dx\; K(x)\right\}$$



INCIDENT WAVE $A e^{ikx}$

TRANSMITTED WAVE $F e^{ikx}$

$$\frac{|F|}{|A|} \sim \exp\left\{ - \int_{x_1}^{x_2} dx \; K(x) \right\}$$

$$\underline{\underline{T \sim \exp\left\{ -2\gamma \right\}}}$$

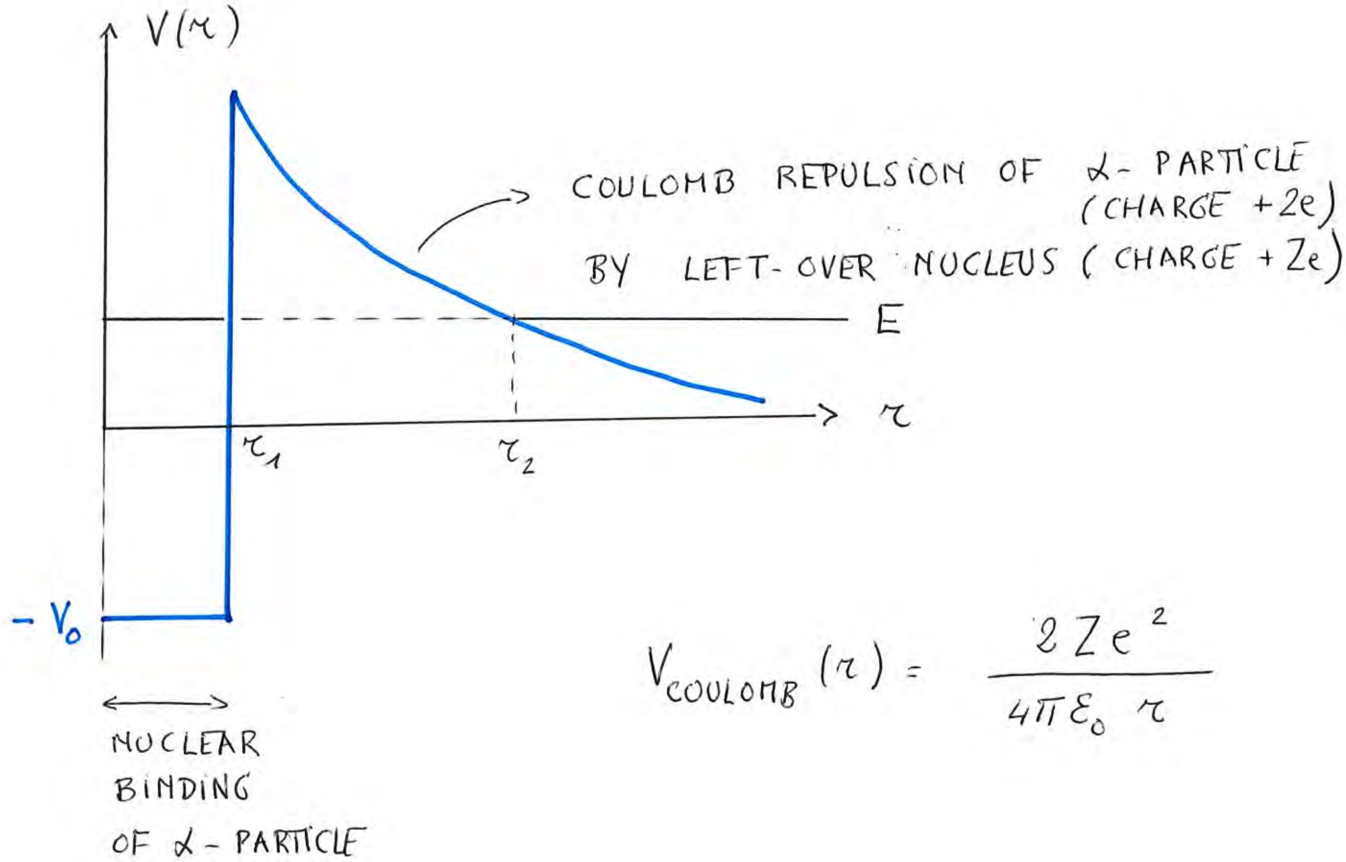
WITH $\boxed{\gamma \equiv \int_{x_1}^{x_2} dx \; K(x)}$

- ## EXAMPLE : GAMOW'S THEORY OF ALPHA DECAY

SPONTANEOUS EMISSION OF α - PARTICLE (2 PROTONS + 2 NEUTRONS)

BY CERTAIN RADIOACTIVE NUCLEI

MODEL POTENTIAL



$$V_{COULOMB}(r) = \frac{2Ze^2}{4\pi\varepsilon_0 r}$$

↳ E : ENERGY OF EMITTED α - PARTICLE : $E = \dfrac{2Ze^2}{4\pi\varepsilon_0 r_2}$

↳ $$\gamma = \int_{r_1}^{r_2} dr \, \frac{1}{\hbar} \sqrt{2m\left[V(r) - E\right]}$$

$$= \frac{\sqrt{2m}}{\hbar} \int_{r_1}^{r_2} dr \sqrt{\frac{2Ze^2}{4\pi\varepsilon_0 r} - E}$$

$$\gamma = \frac{\sqrt{2m}}{\hbar}\int_{r_1}^{r_2} dr \sqrt{\underbrace{\frac{2Ze^2}{4\pi\varepsilon_0 r_2}}_{E}\,\frac{r_2}{r} - E}$$

$$= \frac{\sqrt{2mE}}{\hbar}\int_{r_1}^{r_2} dr \sqrt{\frac{r_2}{r} - 1}$$

SUBSTITUTION $\quad r = r_2 \sin^2\theta$

$$\sqrt{\frac{r_2}{r} - 1} = \frac{\cos\theta}{\sin\theta}$$

$$dr = 2r_2 \sin\theta \cos\theta \, d\theta$$

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\, 2r_2 \int_{\sin^{-1}\sqrt{\frac{r_1}{r_2}}}^{\pi/2} d\theta \underbrace{\cos^2\theta}_{\frac{1}{2}(\cos 2\theta + 1)}$$

$$= \frac{\sqrt{2mE}}{\hbar}\, r_2 \left[\underbrace{\frac{1}{2}\sin 2\theta}_{\sin\theta\cos\theta} + \theta \right]_{\sin^{-1}\sqrt{\frac{r_1}{r_2}}}^{\pi/2}$$

$$= \frac{\sqrt{2mE}}{\hbar}\, r_2 \left[0 - \sqrt{\frac{r_1}{r_2}}\sqrt{1 - \frac{r_1}{r_2}} + \frac{\pi}{2} - \sin^{-1}\sqrt{\frac{r_1}{r_2}} \right]$$

$$= \frac{\sqrt{2mE}}{\hbar} \left[r_2 \left(\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{r_1}{r_2}} \right) - \sqrt{r_1 (r_2 - r_1)} \right]$$

FOR $r_2 >> r_1$ (TYPICAL CASE)

$$\sin^{-1}\sqrt{\frac{r_1}{r_2}} \simeq \sqrt{\frac{r_1}{r_2}}$$

$$\gamma \simeq \frac{\sqrt{2mE}}{\hbar}\left[r_2 \frac{\pi}{2} - 2\sqrt{r_1 r_2} \right]$$

$$E \sim \frac{Z}{r_2} \implies r_2 \sim \frac{Z}{E}$$

$$\boxed{\gamma = K_1 \frac{Z}{\sqrt{E}} - K_2 \sqrt{Z r_1}}$$

WITH K_1, K_2 CONSTANTS

$$K_1 \equiv \left(\frac{e^2}{4\pi\varepsilon_0}\right) \frac{\pi\sqrt{2m}}{\hbar}$$

$$K_2 \equiv \left(\frac{e^2}{4\pi\varepsilon_0}\right)^{1/2} \frac{4\sqrt{m}}{\hbar}$$

↳ PROBABILITY OF ESCAPE OF α - PARTICLE $\sim e^{-2\gamma}$

TIME BETWEEN 2 COLLISIONS WITH WELL $\sim \frac{2r_1}{v}$

v = VELOCITY OF α - PARTICLE

PROBABILITY OF ESCAPE / PER UNIT TIME

$$\sim \left(\frac{v}{2r_1}\right) e^{-2\gamma}$$

LIFETIME $\boxed{\tau \simeq \left(\frac{2r_1}{v}\right) e^{2\gamma}}$

ON LOG PLOT $\ln \tau \sim 2\gamma \sim \frac{1}{\sqrt{E}}$

↳ e.g. FOR $^{238}U \Rightarrow \tau \simeq 4.5\ 10^9$ YEARS !