Lecture 6

We introduced interaction picture

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}$$

$$|\Psi_I(t)\rangle = \underbrace{T}_{\ } \exp\left(-i\int_{t_0}^{t} \hat{H}_{int}(t')\,dt'\right)|\Psi_I(t_0)\rangle$$

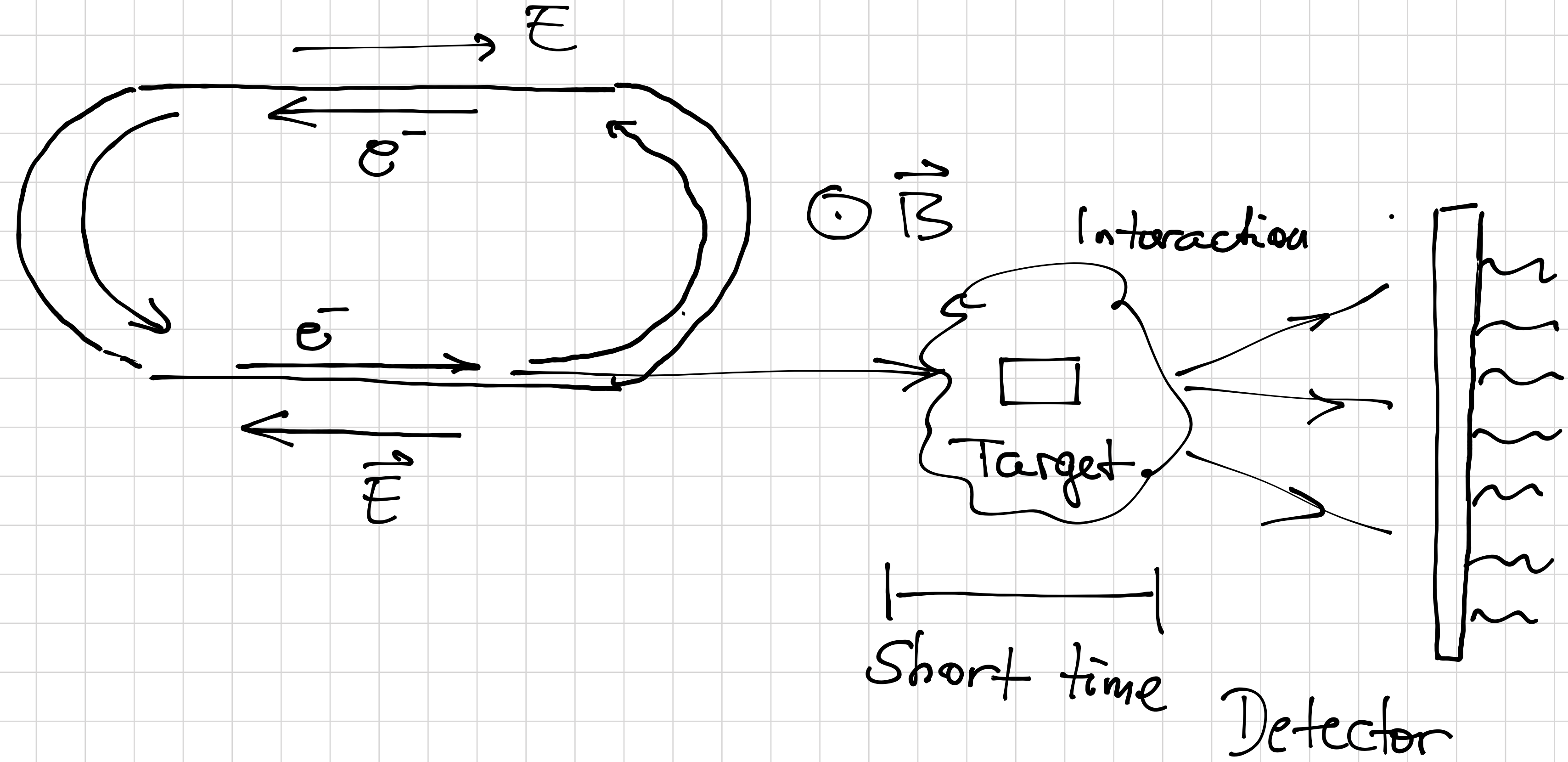
$$\exp(\,.\,.\,.\,) = \mathbb{1} - i\int H_{int} + \frac{(-i)^2}{2}\int\int H\,H \ldots$$

$$T\big(A(t_1)\,B(t_2)\big) = \begin{cases} A(t_1)\,B(t_2)\,, & t_1 > t_2 \\ B(t_2)\,A(t_1)\,, & t_2 > t_1 \end{cases}$$

$$U(t,t_0) = T\exp(\,.\,.\,.\,)$$

We want to study scattering of particles

Δ e^- accelerator



We define "in" and "out" states

$$|i\rangle \qquad\qquad |f\rangle$$
$$t=-\infty \qquad\qquad t=+\infty$$

$|i\rangle, |f\rangle$ = eigenstates of free $\hat{H}_0$

Transition probability $|i\rangle \longrightarrow |f\rangle$

S-matrix : $\lim\limits_{t_\pm \to \pm\infty} \langle f(t_+)|\, U(t_+,t_-)\, |i(t_-)\rangle$

$$= \langle f|\bar{S}|i\rangle = S_{fi}$$

What are these states?

$$|i\rangle = \prod_{i=1}^{n} \left(\sqrt{2\omega_p^i}\, a^+_{i p_i}\right)|0\rangle$$

$$|f\rangle = \prod_{i=1}^{n} \left(\sqrt{2\omega_{p'}^i}\, a^+_{i p_i'}\right)|0\rangle$$

$$S_{fi} \sim \langle f|\, T\left(H_I(x_1) \dots H_I(x_n)\right)|i\rangle$$

$H \sim a^+ a$ (normal ordering)

How are : : and T-ordering related
normal

How to organize the expansion most efficiently?

Consider $T(\phi(x)\phi(y))$

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ipx} + a_p^+ e^{ipx}\right) = \phi^+ + \phi^-$$

$x^0 > y^0$

$$T(\phi(x)\phi(y)) = \left(\phi^+(x) + \phi^-(x)\right)\left(\phi^+(y) + \phi^-(y)\right)$$

$$= \phi^+(x)\,\phi^+(y) + \phi^-(x)\,\phi^+(y)$$
$$+ \phi^-(x)\,\phi^-(y) + \phi^+(x)\,\phi^-(y)$$
$$\sim a \qquad \sim a^+$$
$$= \,:\phi(x)\,\phi(y):\, + [\phi(x), \phi(y)]$$
$$\langle 0| \qquad |0\rangle$$
$$\to D(x-y)$$

$$x^0 < y^0 \hookrightarrow T(\phi(x)\,\phi(y)) = \,:\phi(y)\,\phi(x):$$
$$+ D(y-x)$$

$$\underline{T}(\phi(x)\,\phi(y)) = \,:\phi(x)\,\phi(y):\, + \underbrace{\Delta_F(x-y)}$$
$$\overbrace{\phi(x)\;\phi(y)}$$

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{ik(x-y)}$$

Wick's theorem

$$T(\phi(x_1) \dots \phi(x_n)) = \,:\phi(x_1) \dots \phi(x_n):$$
$$+ \text{ sum of all contractions}$$

Contraction $\dots \overbrace{\phi(x_\ell) \dots \phi(x_m)} \dots$

Insert Feynman propagator for $\phi(x_\ell)\,\phi(x_m)$

The rest of the product of op's unchanged

$\phi \longrightarrow$ real scalar. only 1 type of part.

$\longrightarrow$ all contractions contribute

φ, φ^* complex scalar $\longrightarrow$ particle / antipart.

$$b, b^{\dagger} \qquad c, c^{\dagger} \qquad [b, b^{\dagger}] \neq 0$$

$$[c, c^{\dagger}] \neq 0$$

$$\phi_i \equiv \phi(x_i) \qquad [b, c^{\dagger}] = 0$$

$$T\big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\big) = :\phi_1\phi_2\phi_3\phi_4:$$

$$+ \overline{\phi_1\phi_2} :\phi_3\phi_4: + \overline{\phi_1\phi_3} :\phi_2\phi_4: + \ldots \quad \text{4 similar}$$

$$+ \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \overline{\phi_1\phi_3}\,\overline{\phi_2\phi_4} + \overline{\phi_1\phi_4}\,\overline{\phi_2\phi_3}$$

Proof of Wick's theorem: induction

$n=2$ it is true

For som n $\longrightarrow$ assume it holds

$\hookrightarrow$ need to prove for $n+1$

$$\big(\phi^{+}(x_3) + \phi^{-}(x_3)\big)\Big[:\phi(x_1)\phi(x_2): + \Delta_F(x_1 - x_2)\Big]$$

$\sim a^{\dagger}$

bring to the right of all $a^{\dagger}$

at every time you drag a past $a^{\dagger}$

$\longrightarrow$ you get a contraction

$$\langle f | \mathbb{1} - i \int_{-\infty}^{\infty} H_I(t')\,dt'$$

$$+\frac{(-i)^2}{2}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{t'} \underline{dt''}\, \underbrace{H_I(t')\, H_I(t'')}\, + \ldots \,|\, i\rangle$$

$$\langle f\,|\, S - \mathbb{1}\,|\, i\rangle \qquad\qquad \langle f|\mathbb{1}|i\rangle = \delta_{fi}$$

$$\mathcal{L}_{Yukawa} = \partial_\mu \underset{\text{"nucleon/antinucleon"}}{\underline{\psi}^* \partial^\mu \underline{\psi}} + \underset{\text{"meson"}}{\frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi}$$

$$- M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g\, \psi^* \psi\, \phi$$

$$H_{int} = g\, \psi^* \psi\, \phi$$

b, b^+ : annihilate / create a "nucleon"

c, c^+ : —— " —— "antinucleon"

a, a^+ : —— " —— a meson ϕ

Consider $\psi\psi \longrightarrow \psi\psi$ || ψ nucleon, $\bar{\psi}$ antinucleon

$$|i(p_1, p_2)\rangle = \sqrt{2E_{p_1}}\, \sqrt{2E_{p_2}}\; b^+_{p_1}\, b^+_{p_2}\,|0\rangle$$

$$|f(p_1', p_2')\rangle = \sqrt{2E_{p_1'}}\, \sqrt{2E_{p_2'}}\; b^+_{p_1'}\, b^+_{p_2'}\,|0\rangle$$

$$\langle f\,|\, S - \mathbb{1}\,|\, i\rangle = \langle f\,|\, T(\exp(\underline{-i}\, H_I(t)) - \mathbb{1}\,|i\rangle$$

$$= \langle f\,|\, \frac{(-ig)^2}{2!}\int d^4x_1 \int d^4x_2\, T(\psi_1^+ \psi_1 \phi_1\, \psi_2^* \psi_2 \phi_2)\,|i\rangle$$

$$H_I = g\underline{\psi^*\psi}\phi + \dots$$

$$\rightsquigarrow \psi_{1,2} = \psi(x_{1,2})$$

$$T(\dots) = \underbrace{:\psi_1^+\psi_1\psi_2^+\psi_2:}\ \phi_1\phi_2$$

annihilate 2 nucl. in $|i\rangle$
create 2 nucleons in $|f\rangle$

$$\langle p_1' p_2' | :\psi_1^+\psi_1\psi_2^+\psi_2: | p_1 p_2\rangle$$

our free Hamiltonian is diagonal !

$$\underline{\langle p_1' p_2' | \psi_1^+\psi_2^+ | 0\rangle}\underline{\langle 0 | \psi_1\psi_2 | p_1 p_2\rangle}$$

$$\langle p_1' | \psi_{\underline{1}}^+ | 0\rangle = \underline{e^{+ip_1'x_1}}$$

$$\left(e^{i(p_1'x_1 + p_2'x_2)} + e^{i(p_1'x_2 + p_2'x_1)}\right)$$

$$\cdot\left(e^{-i(p_1x_1 + p_2x_2)} + e^{-i(p_1x_2 + p_2x_1)}\right)$$

$$= e^{ix_1(p_1'-p_1)}e^{ix_2(p_2'-p_2)} +$$

$$+ e^{ix_1(p_1'-p_2)}e^{ix_2(p_2'-p_1)} + (x_1 \leftrightarrow x_2)$$

$$S_{fi}^{(2)} = \frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2\left[e^{ix_1}\dots + (x_1\leftrightarrow x_2)\right]$$

$$\bullet \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{ik(x_1-x_2)}$$

$$\left\{ \int d^4x \, e^{ikx} = (2\pi)^4 \delta^{(4)}(k) \right.$$

$$\Rightarrow (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i\,(2\pi)^8}{k^2-m^2+i\varepsilon}$$

$$\times \Big\{ \underbrace{\delta(p_1'-p_1+k)\,\delta(p_2'-p_2-k)}_{(1)}$$

$$+ \underbrace{\delta(p_2'-p_1+k)\,\delta(p_1'-p_2-k)}_{(2)} \Big\}$$

$$\boxed{= (-ig)^2 \cdot (2\pi)^4 \delta^{(4)}(p_1+p_2-p_1'-p_2') \left[\frac{i}{(p_1'-p_1)^2-m^2+i\varepsilon} + \frac{i}{(p_2'-p_1)^2-m^2+i\varepsilon} \right]}$$

The result at order g^2 in coupling
Most work was needed for trivial $\int$'s

Feynman rules $\to$ pictorial represent. of Wick's theorem

- Draw an external line for each part. in $|i\rangle$, $|f\rangle$
- Mesons $\longrightarrow$ dashed lines - - -

- Nucleons $\rightarrow$ solid lines ———
- To each line assign directed mom. p_i
- An arrow to nucleon/antinucleon

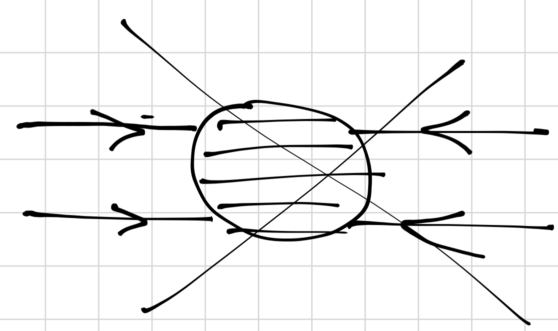
incoming nucleon $|i\rangle \quad \longrightarrow \quad \psi$

antinucleon $|i\rangle \quad \longleftarrow \quad \bar{\psi}$

outgoing $|f\rangle \quad \longrightarrow \quad \psi$

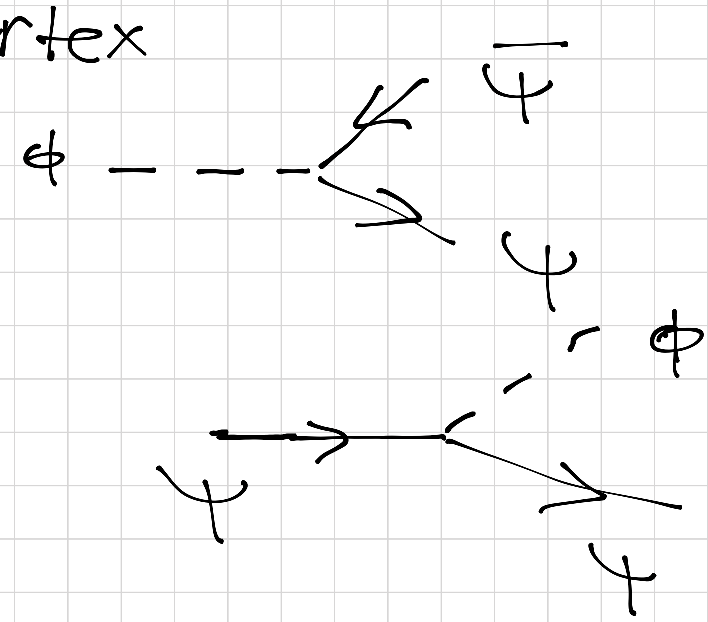
$|f\rangle \quad \longleftarrow \quad \bar{\psi}$

$\psi\psi \rightarrow \bar{\psi}\bar{\psi}$



- Interaction vertex

$g\psi^*\psi$



Each diagram exactly corresponds to a term in expansion of $S - \mathbb{1}$

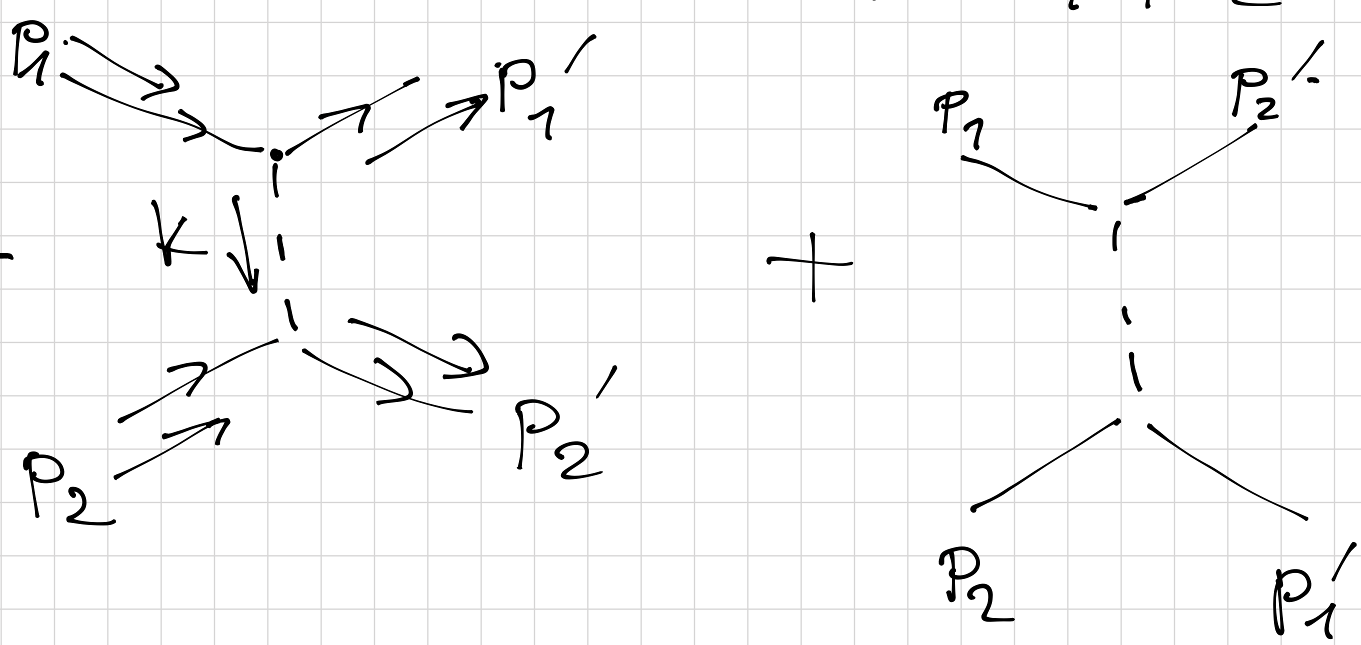
To each vertex: $(-ig)(2\pi)^4 \delta^{(4)}(\Sigma_i k_i)$

$\delta^{(4)}(k_1+k_2+k_3)$

To each internal line assign

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \quad \text{(meson)}$$

$$\text{or} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i\varepsilon} \quad \text{(nucleon/anti)}$$



$$e^{-i\alpha} = 1 - i\alpha + \frac{(-i\alpha)^2}{2!} + \dots \frac{(-i\alpha)^n}{n!}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} (-ig)^2$$

$$(2\pi)^4 \delta(p_1 - p_1' - k) \quad (2\pi)^4 \delta(p_2 - p_2' + k)$$