Lecture 5

We introduced propagator

$$D(x-y) = \langle 0|\phi(x)\,\phi(y)|0\rangle$$
$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{e^{-ip(x-y)}}{2\sqrt{\vec{p}^{\,2}+m^2}}$$

Causality: for any $(x-y)^2 = (x^0-y^0)^2 - (\vec{x}-\vec{y})^2 < 0$

$$\Delta(x-y) = D(x-y) - D(y-x) = 0$$

Introduced Feynman propagator

$$\Delta_F = \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & y^0 > x^0 \end{cases}$$

We will see that $\Delta_F \to$ central role in QFT interaction

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Now $\quad D(x-y) - D(y-x) = 0 \quad$ for $(x-y)^2 < 0$

$D(x-y) \neq 0 \qquad$ choose $x^0 = y^0$

What is the meaning

$$D(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{e^{+i\vec{p}(\vec{x}-\vec{y})}}{2\sqrt{\vec{p}^{\,2}+m^2}} \qquad |\vec{x}-\vec{y}| = r \qquad p_z \,//\, \vec{x}-\vec{y}$$

$$= \int_0^\infty \frac{p^2\,dp}{(2\pi)^3}\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta\,\frac{e^{ipr\cos\theta}}{2\sqrt{p^2+m^2}}$$

$$= \frac{-i}{8\pi^2 r} \int_0^\infty \frac{p\,dp}{\sqrt{p^2+m^2}} \left(e^{ipr} - e^{-ipr}\right)$$

$\int \longrightarrow$ complex plane

$e^{ipr} \longrightarrow$ upper plane $(\operatorname{Im} p > 0,\ e^{-\operatorname{Im} p\, r})$

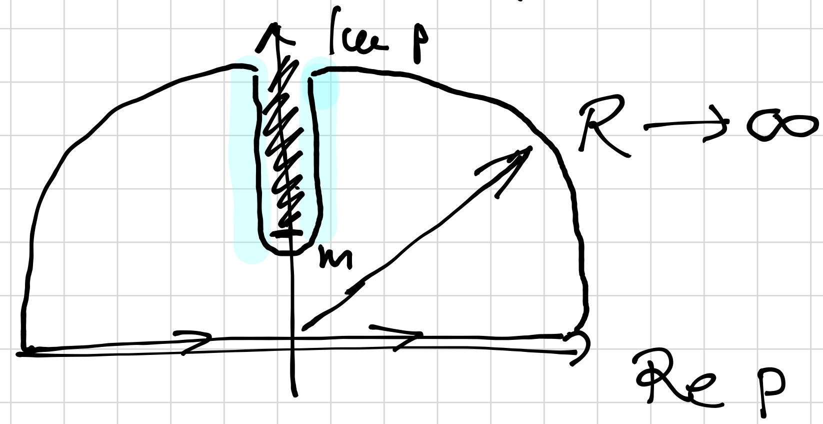
$e^{-ipr} \longrightarrow$ lower $\longrightarrow$ $\operatorname{Im} p < 0$

Attn: for $p = im$, $|p| > m$

$$p^2 + m^2 < 0$$

$$\sqrt{p^2+m^2} = \pm i \sqrt{|p^2+m^2|}$$

1st



$$\oint = \int_0^\infty \ldots + \int_R \ldots + \int_{\text{Cut}}$$

$$\int_R : \quad p = R \cdot e^{i\varphi} \Longrightarrow \int_R = \int_0^\pi \frac{R^2 e^{2i\varphi}\, d\varphi}{\sqrt{R^2 e^{2i\varphi} + m^2}} \to 0$$

$$e^{iRr\cos\varphi}\; e^{-Rr\sin\varphi} \xrightarrow[R\to\infty]{} \longrightarrow 0$$

$$\int_{\text{cut}} : = \int_{i\infty+\varepsilon}^{im+\varepsilon} + \int_{im-\varepsilon}^{i\infty-\varepsilon} \qquad p \longrightarrow iy$$

$$\sqrt{(i|p| \pm \varepsilon)^2 + m^2} = \pm i \sqrt{|p|^2 - m^2}$$

$$\int_{\text{cut}} = \frac{2}{8\pi^2 r} \int_m^\infty \frac{y\,dy}{\sqrt{y^2-m^2}} e^{-yr} \quad \times 2$$

$\oint = 0$: no singularities inside

$$D(r) \sim \frac{1}{r} \int_1^\infty \frac{x\,dx}{\sqrt{x^2-1}} e^{-mrx} \qquad y = m \cdot x$$

$K_1(mr)$

Asymptotic behavior $D(r)$ $r \to \infty$

$r \to \infty$ $\quad D(r) \sim \frac{e^{-mr}}{r} \to$ Yukawa p. (short-range)

$m = 0$ $\quad D(r) \sim \frac{1}{r} \longrightarrow$ Electrostatic Gravity

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Field leaks outside the lightcone

Classical p. $\sim \frac{1}{r} \longrightarrow$ massless field

Short-range (strong, weak) $\sim \frac{e^{-mr}}{r}$ massive field

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More to this $\to$ next topic

Before going there : nonrelativistic limit

K-G eq. $\left(\partial_\mu \partial^\mu + m^2\right)\varphi = 0$

$$\varphi(\vec{x}, t) = \underline{e^{-imt}}\,\psi(\vec{x}, t)$$

$$\ddot{\varphi} - \vec{\nabla}^2 \varphi + m^2 \varphi = \cancel{\ddot{\psi}} - 2im\dot{\psi} - \vec{\nabla}^2 \psi = 0$$

Non-rel. limit $|\vec{p}| \ll m$

$\hookrightarrow |\ddot{\psi}| \ll m|\dot{\psi}|$

slowly - varying field

$$i\frac{\partial \psi}{\partial t} = -\frac{\vec{\nabla}}{2m}\psi$$

$$\mathcal{L} = i\psi^* \dot{\psi} - \frac{1}{2m}\vec{\nabla}\psi^* \vec{\nabla}\psi$$

Symmetry $\psi \to e^{i\alpha}\psi$ $\qquad \psi \to \psi + i\alpha\psi$

$\qquad \psi^* \to \psi^* - i\alpha\psi^*$

$\hookrightarrow$ Conserved Noether current

$Q = \int d^3\vec{x}\, j^0$ $\qquad j^\mu = \left(-\psi^*\psi,\ \frac{i}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)\right)$

$\to \quad Q = \int d^3\vec{x}\ :\psi^*\psi: \ = \text{constant}$

$\hookrightarrow$ particle number

- conserved # of particles
- complex field but only 1 kind (no antiparticles)

$$\psi^*(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\, a_p^\dagger\, e^{-i\vec{p}\vec{x}}$$

$$|\vec{x}\rangle = \psi^*(\vec{x})|0\rangle$$

$$\hat{X}^i = \int d^3\vec{x}\, \vec{x}\, \psi^*\psi$$

Superposition of 1-particle states $\quad |\varphi\rangle = \int d^3\vec{x}\, \varphi(x)\, |\vec{x}\rangle$

$$\underset{\sim}{P}^i |\varphi\rangle = \int d^3\vec{x} \left(-i \frac{\partial \varphi}{\partial x^i}\right) |\vec{x}\rangle$$

$$i \frac{\partial \varphi}{\partial t} = -\frac{\vec{\nabla}^2}{2m} \varphi$$

$$Q = \int d^3\vec{x} \, |\varphi(\vec{x})|^2$$

$|\varphi(\vec{x})| \rightarrow$ Schr. WF with statistical interpretation

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Interacting fields

We diagonalized the Hamiltonian in normal modes $\hat{a}_{\underline{p}}, \hat{a}^+_{\underline{p}}$;

Every mode evolves indipendently

$\hookrightarrow$ free theory $\rightarrow$ $\mathcal{L}$ quadratic in ϕ

$\Longrightarrow$ Interaction $=$ Higher orders in ϕ

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi - \sum_{n \geq 3} \frac{\lambda_n}{n!} \phi^n$$

λ_n coupling constants

$[S] = \hbar = 0$ (means E°)

$S = \int d^4x \, \mathcal{L} \Rightarrow [\mathcal{L}] = 4$

$[\partial_\mu] = 1 \quad [\phi] = 1$

$\Rightarrow [\lambda_n] = 4 - n$

$[\lambda_3] = 1 \Rightarrow$ "relevant interaction"

High energies: $\frac{\lambda_3}{E} \ll$

large at low E

$[\lambda_4] = 0 \Rightarrow$ "marginal"

$|\lambda_4| \ll 1$

$\hookrightarrow$ small effects

$[\lambda_{n\geq 5}] = n-4 \Rightarrow$ "irrelevant"

$\lambda_n \sim \frac{E^{n-4}}{\Lambda^{n-4}}$

low energies $\ll$

blows up at large E

The power of dimensional analysis

Imagine we extend our K-G theory to very high scale Λ $[\Lambda] = 1$

What will it look at $E \ll \Lambda$?

$$\mathcal{L}_0 - \sum_{n\geq 5} \frac{1}{n!} g_n \frac{E^{n-4}}{\Lambda^{n-4}} \ldots$$

$\Lambda \to \infty$

Interaction $\to 0$

Weak interactions

$\hookrightarrow$ Fermi int.

$$\frac{G_F}{2\sqrt{2}} \bar{\psi} \gamma^\mu (1-\gamma_5) \psi \; \bar{\psi} \gamma_\mu (1-\gamma_5) \psi$$

$$[G_F] = -2 \quad \left(\frac{1}{\Lambda^2}\right)$$

$G_F \sim 1 \cdot 10^{-5}\ GeV^{-2}$

$\hookrightarrow$ discovery of $W^\pm$ with $m_W \approx 80$ GeV

$$G_F \sim \frac{4\pi\alpha}{M_Z^2}$$

Z^0 with $m_Z \approx 90$ GeV

How to solve QFT w. interactions?

1. Perturbation theory
   $\hookrightarrow$ Feynman diagrams   infinite series
2. Variational methods
3. Monte Carlo simulations (eg. on discretized space-time $\to$ lattice)
4. Exact solutions
   1+1 dim (Bethe ansatz)
   2+1 dim Topological QFT
   Large N_c limit

Perturbation theory

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}$$

$H = T + V$

$L = T - V$

t-dep. Schrodinger eq. $i\frac{d}{dt}|\psi_s\rangle = \hat{H}|\psi_s\rangle$

Interaction picture

All states and operators are transformed as

$$|\Psi_I(t)\rangle = e^{i\hat{H}_0 t}|\Psi_s(t)\rangle$$

$$O_I = e^{i\hat{H}_0 t} O_s e^{-i\hat{H}_0 t}$$

$$i\frac{d}{dt}|\Psi_I(t)\rangle = -\hat{H}_0 e^{i\hat{H}_0 t}|\Psi_s\rangle + e^{i\hat{H}_0 t}\underbrace{i\frac{d}{dt}|\Psi_s\rangle}_{= \hat{H}|\Psi_s\rangle}$$

$$= \left(\cancel{-\hat{H}_0 e^{i\hat{H}_0 t}} + \underbrace{e^{i\hat{H}_0 t}(\cancel{\hat{H}_0} + \hat{H}_{int})e^{-i\hat{H}_0 t}}_{= (\hat{H}_{int})_I(t)} e^{i\hat{H}_0 t}\right)|\Psi_s\rangle$$

$$i\frac{d}{dt}|\Psi_I(t)\rangle = \underline{\hat{H}_{int}(t)}\,|\Psi_I(t)\rangle$$

Propagator $\hat{U}(t,t_0)$: $|\Psi_I(t)\rangle = \hat{U}(t,t_0)|\Psi_I(t_0)\rangle$

$$i\frac{d}{dt}\hat{U}(t,t_0) = \hat{H}_{int}(t)\,\hat{U}(t,t_0)$$

$$U(t,t_0) \cancel{=} e^{-i\int_{t_0}^{t}\hat{H}_{int}(t')\,dt'}$$

$$\mathrm{Exp}[Op.] = \mathbb{1} + Op + \frac{1}{2}(Op)^2 \ldots$$

$$\frac{(-i)^2}{2}\left(\int_{t_0}^{t}\hat{H}_{int}(t')\,dt'\right)^2$$

Back into diff. eq.

$$\frac{d\hat{U}}{dt} = \ldots -\frac{1}{2}\left[\int_{t_0}^{t} H_{int}(t')dt'\right] H_{int.}(t)$$

$$-\frac{1}{2} H_{int}(t) \left[\text{—}\,''\,\text{—} \right] + \ldots$$

1^{st} term not ok.

$$[H(t), H(t')] \neq 0 \quad t \neq t'$$

Correct solution:

$$\hat{U}(t, t_0) = T\exp\left(-i \int_{t_0}^{t} H_{int}(t')dt'\right)$$

$$T\left(O_1(t_1)\, O_2(t_2)\right) = \begin{cases} O_1(t_1)\, O_2(t_2), & t_1 > t_2 \\ O_2(t_2)\, O_1(t_1), & t_2 > t_1 \end{cases}$$

$$i\frac{\partial}{\partial t} T\left(\exp\left(-i \int_{t_0}^{t} \underline{H_I(t')dt'}\right)\right)$$

$$= T\left[H_I(t) \exp\left(-i \int_{t_0}^{t} H_I(t')dt'\right)\right]$$

Trivial proof

<u>Dyson formula</u>