Where we arrived at:

$$\phi = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[\hat{a}_p e^{-i\underbrace{px}_{p_\mu x^\mu}} + \hat{a}_p^+ e^{ipx} \right]$$

$$\begin{cases} \hat{H}_{KG} = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_p a_p^+ a_p & \text{from } \int d^3x\, T^{00} \\ \hat{P}^i = \int \frac{d^3\vec{p}}{(2\pi)^3} p^i a_p^+ a_p & \text{from } \int d^3x\, T^{0i} \end{cases}$$

<u>Exercise</u>

$\rightarrow \hat{P}^\mu$

$$\{p_i, p_j\}_{PB} = 0 \rightarrow [\hat{P}^\mu, \hat{P}^\nu] = 0$$

→ Identify <u>all</u> conserved currents from Noether theorem

→ "Put hats" on all of them

→ Check that all commutations are obeyed

⟹ Demonstrated that we found a unitary representation of the Poincare group

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$$\hat{H}_{KG} = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_p a_p^+ a_p$$

Classical

$\omega(a^+a + \underset{\sim}{\tfrac{1}{2}})$

$$\text{QFT}: \int_{\dots} \omega_p \left[a_p^\dagger a_p + \underbrace{\tfrac{1}{2}(2\pi)^3 \delta^3(0)}_{\to \text{ Vacuum energy}} \right]$$

There is no way to measure it $\to$ only difference

There is gravity that feels it

<u>GR</u>

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

Cosmological constant $\Lambda = \frac{E_0}{V}$

Astrophysics: 70% of Energy density in Universe is $\Lambda \sim (10^{-3} eV)^4$

Standard Moder is valid from $\sim 10^{-2} eV$ to TeV

Next step: construct the Fock space

Vacuum: $|0\rangle$ for any p $\hat{a}_p |0\rangle = 0$

1 particle $\to$ Hilbert space $\mathcal{H}$

n particles $\to$ Fock space $\mathcal{F} = \oplus_n \mathcal{H}$

$$|p_1 \, p_2 \dots p_n\rangle = \hat{a}^\dagger_{p_1} \hat{a}^\dagger_{p_2} \dots \hat{a}^\dagger_{p_n} |0\rangle$$

1 particle basis $|p\rangle$

$$\mathbb{1}_{1p} = \int \frac{d^3 \vec{p}}{(2\pi)^3} |p\rangle\langle p| \cdot N_p$$

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} |p\rangle \underbrace{\langle p | k \rangle}_{(2\pi)^3 \delta^3(p-k)} = |k\rangle$$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} = \int \frac{d^4 p}{(2\pi)^3} \cdot \underbrace{\delta(p^{0\,2} - \vec{p}^{\,2} - m^2)}_{\delta(p^{0\,2}-\omega_p^2) = \frac{1}{2\omega_p}\delta(p^0-\omega_p)} \cdot \underline{2\omega_p}$$

$$|p\rangle = \sqrt{2\omega_p}\, \hat{a}_p^{+} |0\rangle$$

$$\hat{p}^{\mu} |0\rangle = 0 \qquad \hat{p}^{\mu} |p_1 \dots p_n\rangle = \left(\sum_{i=1}^{n} p_i^{\mu}\right) |p_1 \dots p_n\rangle$$

We concluded the quantization of free real K-G field

$$\hat{H}_{KG} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \omega_p a_p^{+} a_p$$

$$\hat{N} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, a_p^{+} a_p$$

$$[\hat{H}, \hat{N}] = 0$$

$$\Downarrow$$

$$\hat{N} |p_1 \dots p_n\rangle = n |p_1 \dots p_n\rangle$$

$$\frac{dn}{dt} = 0$$

Complex field ?

$$\mathcal{L} = (\partial_\mu \varphi^{+})(\partial^\mu \varphi) - m^2 \varphi^{+} \varphi$$

$$\hookrightarrow (\partial_\mu^2 + m^2)\varphi = (\partial_\mu^2 + m^2)\varphi^{+} = 0$$

$\longrightarrow$ 2 types of creation / annihilation op.

b_p, b_p^{+} ; c_p, c_p^{+}

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p e^{-ipx} + c_p^+ e^{ipx} \right)$$

$$\psi^+(x) = \text{———}\,''\,\text{—} \left(b_p^+ e^{ipx} + c_p e^{-ipx} \right)$$

$\downarrow$

conjugate fields $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \dot{\psi}^+$

$\pi^+ = \dot{\psi}$

ETCR $[b_p, b_q^+] = [c_p, c_q^+] = (2\pi)^3 \delta^3(\vec{p}-\vec{q})$

$\emptyset$ otherwise

$\longrightarrow$ particle + antiparticle

This theory has a conserved charge

$$Q = i \int d^3\vec{x} \left(\pi \psi - \pi^+ \psi^+ \right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left(c_p^+ c_p - b_p^+ b_p \right) = \underline{n_c - n_b}$$

$\updownarrow$

$\underline{\psi \to e^{i\alpha} \psi}$

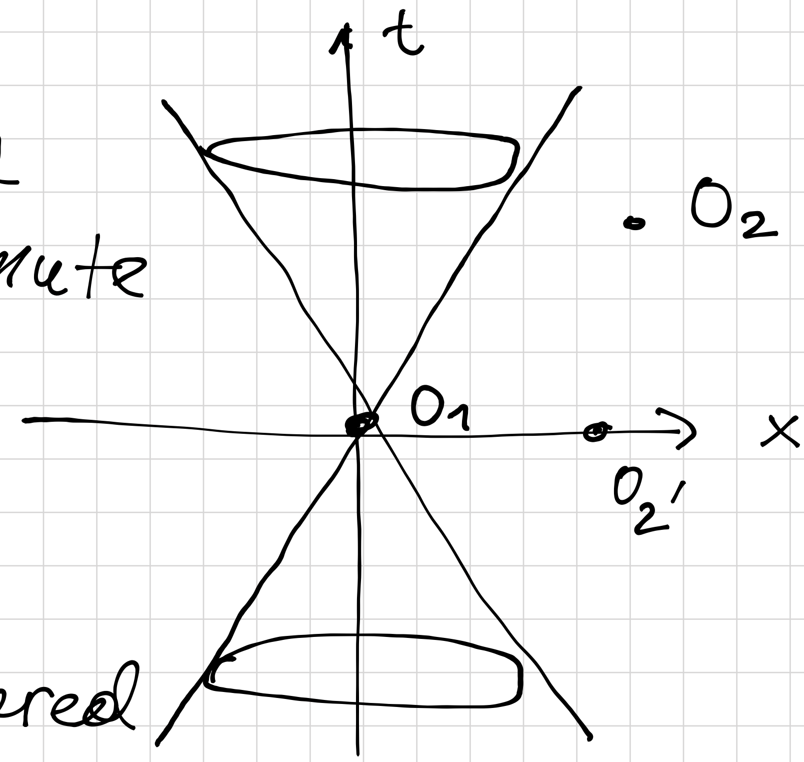
Trivial in classical theory (n_c and n_b conserved)

In QFT such a conservation law even with interactions! ($n_c, n_b \neq$ const.)

$$\boxed{n_c - n_b = \text{const.}}$$

Causality

All spacelike - separated operators should commute



$$[O_1(x), O_2(y)] = 0$$

for all $(x-y)^2 < 0$

[Until now we considered
ETCR $\longrightarrow$ space-time separations.

$$\Delta(x-y) = [\phi(x), \phi(y)]$$

$$\Downarrow$$

$$\Delta(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]$$

Importantly : c - number

Properties: • Lorentz - invariant

• Non-zero for time-like separ.

• Zero for space-like separation

Choose $x^0 = y^0$ ETCR $[\phi(x), \phi(y)]_{x^0=y^0} = 0$

Timelike : choose $\vec{x} = \vec{y}$

$$\Delta(x-y)\Big|_{t\to\infty} \sim e^{-imt} - e^{+imt}$$

$$\neq 0$$

K-G QFT is causal : commutators do vanish outside the lightcone

Prepare a particle at a point x

→ what is the amplitude of finding it at a point y ?

$$\doteq \langle 0 | \phi(x)\, \phi(y) | 0 \rangle = \int \frac{d^3\vec{p}\, d^3\vec{p}\,'}{(2\pi)^6} \frac{1}{\sqrt{4\omega_p \omega_{p'}}}$$

$$\times \langle 0 | \underbrace{\hat{a}_p \hat{a}_{p'}^{+}}_{[\hat{a}_p, \hat{a}_{p'}^{+}] + \cancel{\hat{a}_{p'}^{+} a_p}} | 0 \rangle \, e \; e^{-ipx + ip'y}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip(x-y)} \equiv D(x-y)$$

Propagator

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \underline{D(x-y)} - \underline{D(y-x)}$$

For $(x-y)^2 < 0$ $D(x-y) \neq 0$ by $[\,,\,] = 0$

The amplitude for particle to travel $x \to y$ cancels that for $y \to x$

For complex field particle $x \to y$

anti-particle $y \to x$

(One of) central quantities in QFT

Feynman propagator

$$\Delta_F(x-y) = \langle 0 | T\, \phi(x)\, \phi(y) | 0 \rangle$$

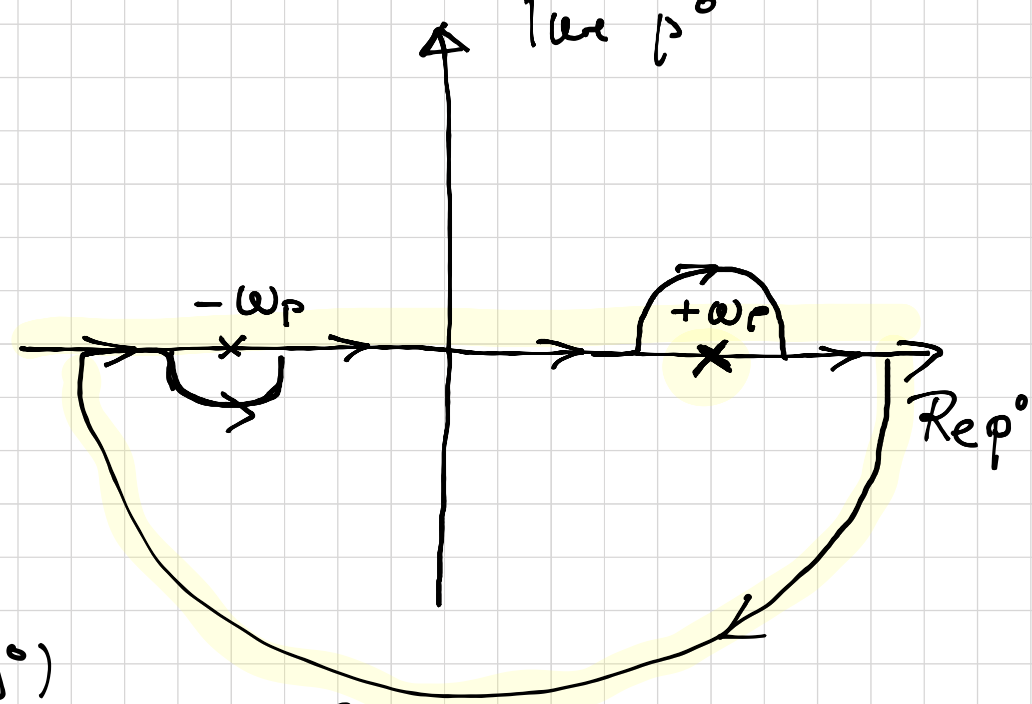
$$= \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & y^0 > x^0 \end{cases}$$

Claim $\boxed{\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip(x-y)}}$

2 poles : $p^0 = \pm\omega_p = \pm\sqrt{\vec{p}^2+m^2}$

$\int d^4p = \int_{-\infty}^{\infty} dp^0 \int d^3\vec{p}$

1. $x^0 > y^0$



$e^{-i \cdot i \, Im\, p^0 |x^0-y^0|} \longrightarrow 0$

$Im\, p^0 < 0$

$$\int_{-\infty}^{\infty} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p)(p^0+\omega_p)} = \oint dp^0 \text{ — '' — }$$

$$= 2\pi i \sum \text{residues}$$

$$= 2\pi i \, (-1) \, \frac{e^{-i\omega_p(x^0-y^0)}}{2\omega_p}$$

The inside of C
is on the right (clockwise)

$$\Delta_F(x-y)\Big|_{x^0-y^0} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip(x-y)} = D(x-y)$$

$x^0-y^0 < 0 \longrightarrow$ need to close the circle into the upper half plane

$$\Delta_F(x-y)\big|_{y^0-x^0>0} = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\omega_p} e^{-i\omega_p(y^0-x^0)-i\vec{p}(\vec{y}-\vec{x})}$$

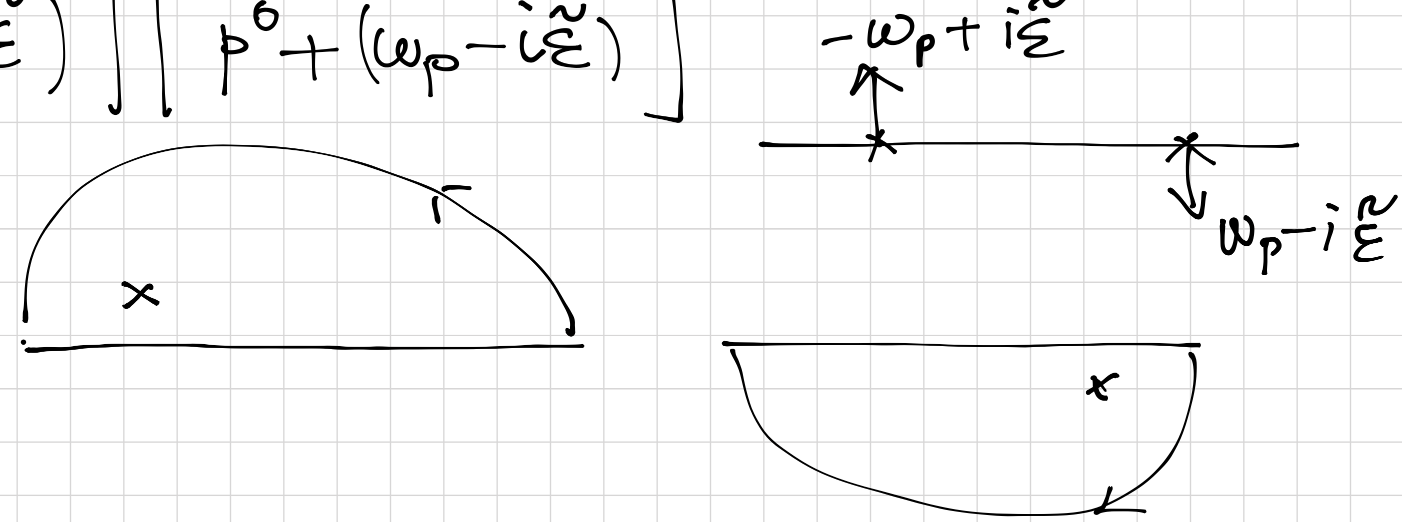
$$d^3\vec{p} \longrightarrow d^3(-\vec{p}) = d^3\vec{p}$$

$$\Delta_F(x-y)\big|_{y^0>x^0} = D(y-x)$$

Proved

"$+i\varepsilon$ prescription" $\quad \dfrac{1}{p^2-m^2} \longrightarrow \dfrac{1}{p^2-m^2+i\varepsilon}$

$$\left[p^0-(\omega_p-i\tilde{\varepsilon})\right]\left[p^0+(\omega_p-i\tilde{\varepsilon})\right]$$



Math

Feynman propagator =

= Green's function of K-G functional

$$(\partial_\mu\partial^\mu+m^2)\,i\Delta_F(x-y) = \delta^{(4)}(x-y)$$

$$\int\frac{d^4p}{(2\pi)^4}\frac{i^2(\partial_\mu^2+m^2)e^{-ipx}}{p^2-m^2+i\varepsilon} = \int\frac{d^4p}{(2\pi)^4}\frac{-\cancel{(-p^2+m^2)}e^{\cdots}}{\cancel{p^2-m^2}}$$

$$= \delta^{(4)}(x-y)$$

$$L\,u(x) = f(x)$$

$$L\,G(x,s) = \delta(x-s) \implies u(x) = u_0(x) + \int G(x,s)f(s)\,ds$$

$$L u_0 = 0$$