→ Classical Lagrangians
→ Conserved currents ← Symmetries
(Noether Theorem)

Energy-momentum tensor
(translations)

4D Angular momentum
(infinitesimal Lorentz tr.)

→ Hamiltonian formalism
→ conjugate coordinates φ, $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$
→ Poisson brackets

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q_i, p_i $\qquad \{q_i, p_j\} = \delta_{ij}$

"Put the hats on"

Commutators $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

$\mathcal{H}(\varphi, \pi)$ $\qquad \varphi(\vec{x}) \qquad \pi(\vec{x})$

→ commutation relations

$$[\varphi_a(\vec{x}), \pi_b(\vec{y})] = i\delta^3(\vec{x}-\vec{y})\delta_{ab}$$

$$[\varphi_a, \varphi_b] = [\pi_a, \pi_b] = 0$$

System described by "WF" φ

$$i \frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle$$

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$|\Psi\rangle$ accounts for ∞ number of field configurations

Usually to solve our system we diagonalise H

$\hookrightarrow$ Energy spectrum

∞ # degrees of freedom (coupled)

// Free theories: d.o.f. decouple!

Consider K-G equation

$$\rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

$$\phi = \phi(t, \vec{x})$$

$$H = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + m^2\phi^2$$

$$\phi(t, \vec{p}) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\vec{x}} \phi(t, \vec{x})$$

$$\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \rightarrow \frac{\partial^2}{\partial t^2} + \vec{p}^2$$

$$\left[\frac{\partial^2}{\partial t^2} + (\vec{p}^2 + m^2)\right] \phi(t, \vec{p}) = 0$$

$=$ Harmonic oscillator with $\omega_p = \sqrt{\vec{p}^2 + m^2}$

To quantize $\phi(t,\vec{x})$ we need to quantize an infinite number of H.O. (3)

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2 \qquad [\hat{q},\hat{p}] = i$$

$\downarrow$

Ladder operators (raising / lowering)

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right)$$
$$\hat{a}^+ = \sqrt{\frac{m\omega}{2}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right)$$
$$[\hat{a},\hat{a}^+] = 1$$

$\Downarrow$

$$\rightarrow \hat{H} = \omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

$$[\hat{a},\hat{H}] = \omega\left[\hat{a}\hat{a}^+\hat{a} - \hat{a}^+\hat{a}\hat{a}\right] = \omega[\hat{a},\hat{a}^+]\hat{a} = \omega\hat{a}$$

$$[\hat{a}^+,\hat{H}] = -\omega\hat{a}^+$$

$$i\frac{d}{dt}\hat{a}(t) = [\hat{a},\hat{H}] \rightarrow$$

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$
$$\hat{a}^+(t) = \hat{a}^+(0)e^{+i\omega t}$$

Number operator $\hat{N} = \hat{a}^+a$

$$\hat{N}|n\rangle = n|n\rangle$$

$$\hat{H}|n\rangle = \left(n+\frac{1}{2}\right)\omega|n\rangle$$

→ We interpret the state $|n\rangle$ as n particles in the mode $\sim \omega_p$ (4)

1 HO → Hilbert space

n particles → Fock space $\equiv \oplus_n \mathcal{H}_p$

$$|n\rangle = \mathcal{N} (\hat{a}^+)^n |0\rangle$$

Vacuum state : $|0\rangle : \hat{a}_p |0\rangle = 0$

$$\hat{a}_p^+ \hat{a}_p |0\rangle = 0$$

$$\langle n | n \rangle \Rightarrow \mathcal{N} \frac{1}{\sqrt{(n+1)!}}$$

$$\phi(t, \vec{x}) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \left[\hat{a}_p(t) e^{i\vec{p}\vec{x}} + \hat{a}_p^+(t) e^{-i\vec{p}\vec{x}} \right]$$

$$\begin{cases} \hat{a}_p(t) = \hat{a}(0) e^{-i\omega t} \\ \hat{a}_p^+(t) = \hat{a}^+(0) e^{+i\omega t} \end{cases}$$

Commutation relations

$$\pi(t, \vec{x}) \overset{KG}{=} \dot{\phi}(t, \vec{x}) = -i \int \frac{d^3 \vec{p}}{(2\pi)^3} \omega_p \left[\hat{a} e^{i\vec{p}\vec{x}} - \hat{a}^+ e^{-i\vec{p}\vec{x}} \right]$$

→ Check explicitly

$$[\hat{a}_p, \hat{a}_q] = [\hat{a}_p^+, \hat{a}_q^+] = 0$$

$$[\hat{a}_p, \hat{a}_q^+] = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

Continuous description

$$\begin{cases} [\phi(t,\vec{x}),\ \phi(t,\vec{y})] = [\pi(t,\vec{x}),\pi(t,\vec{y})] = 0 \\ [\phi(t,\vec{x}),\ \pi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y}) \end{cases}$$

Equal time commutation relations

↳ Exercise (use CR for $\hat{a}, \hat{a}^+$)

↳ Normalization to be checked!

$$\phi = \int \frac{d^3\vec{p}}{(2\pi)^3} N_p \left(\hat{a}_p e^{-ip_\mu x^\mu} + \hat{a}_p^+ e^{+ipx}\right)$$

$$\pi = -i\int \frac{d^3\vec{p}}{(2\pi)^3} N_p \omega_p \left(\hat{a}_p e^{-ipx} - \hat{a}_p^+ e^{ipx}\right)$$

To determine N_p:

$$[\phi(x),\pi(y)] = \left[\int \frac{d^3\vec{p}}{(2\pi)^3} N_p \left(a_p e^{-ipx} + a_p^+ e^{ipx}\right),\right.$$
$$\left. -i\int \frac{d^3\vec{q}}{(2\pi)^3} N_q \omega_q \left(a_q e^{-iqy} - a_q^+ e^{iqy}\right)\right]$$

Use
$$\begin{aligned} &[a_p, a_q] = [a_p^+, a_q^+] = 0 \\ &[a_p, a_q^+] = (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \end{aligned}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{q}}{(2\pi)^3} N_p N_q \omega_q \Big\{ i[a_p, a_q^+] e^{-ipx+iqy}$$
$$- i[a_p^+, a_q] e^{ipx-iqy}\Big\}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \, i\omega_p N_p^2 \left(e^{-ip(x-y)} + e^{+ip(x-y)}\right)$$

ETCR $\rightarrow$ $x^0 = y^0 \Rightarrow (\ldots) = e^{i\vec{p}(\vec{x}-\vec{y})} + e^{-i\vec{p}(\vec{x}-\vec{y})}$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{-i\vec{p}(\vec{x}-\vec{y})} = \delta^3(\vec{x}-\vec{y})$$

$$\Downarrow$$

For $[\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$ to hold

$N_p = \frac{1}{\sqrt{2\omega_p}}$ has to hold

$$\Downarrow$$

$$\boxed{\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[\hat{a}_p e^{-ipx} + \hat{a}_p^+ e^{ipx}\right]}$$

Now we can rewrite our Hamiltonian:

$$H = \int d^3\vec{x} \, \mathcal{H} = \int d^3\vec{x} \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right)$$

$$= \frac{1}{2}\int \frac{d^3\vec{x}\, d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6}$$

$$\times \left[-\frac{\sqrt{\omega_p\omega_q}}{2}\left(a_p e^{i\bar{p}\bar{x}} - a_p^+ e^{-i\bar{p}\bar{x}}\right)\left(a_q e^{i\bar{q}\bar{x}} - a_q^+ e^{-i\bar{q}\bar{x}}\right)\right.$$

$$+\frac{1}{2\sqrt{\omega_p\omega_q}}\left(i\vec{p}\,a_p e^{i\vec{p}\vec{x}} - i\vec{p}\,a_p^+ e^{-ipx}\right)$$

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$$\cdot\left(i\vec{q}\,a_q e^{i\vec{q}\vec{x}} - i\vec{q}\,a_q^+ e^{-i\vec{q}\vec{x}}\right)$$

$$+\frac{m^2}{2\sqrt{\omega_p\omega_q}}\left(a_p e^{i\vec{p}\vec{x}} + a_p^+ e^{-i\vec{p}\vec{x}}\right)\left(a_q e^{i\vec{q}\vec{x}} + a_q^+ e^{-i\vec{q}\vec{x}}\right)\Big]$$

Intermediate step (example)

$$(a_p a_q): \int\frac{d^3\vec{x}}{(2\pi)^3} e^{i\vec{x}(\vec{p}+\vec{q})}\left(-\frac{\sqrt{\omega_p\omega_q}}{2} + \frac{\vec{p}\vec{q}+m^2}{2\sqrt{\omega_p\omega_q}}\right)$$

$$= \delta^3(\vec{p}+\vec{q})\cdot\frac{-\omega_p^2+\vec{p}^2+m^2}{2\omega_p}$$

$\underline{\omega_{-p}=\omega_p}$

$$\hookrightarrow H = \frac{1}{4}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\omega_p}\Big[\underbrace{\cancel{(-\omega_p^2+\vec{p}^2+m^2)}}_{\to 0}(a_p a_{-p} + a_p^+ a_{-p}^+)$$

$$+(\omega_p^2+\vec{p}^2+m^2)(a_p a_p^+ + a_p^+ a_p)\Big]$$

$\omega_p^2 = \vec{p}^2+m^2 \longrightarrow 1^{st}$ term vanishes

$$\Downarrow$$

$$H = \frac{1}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\,\omega_p\left[a_p a_p^+ + a_p^+ a_p\right]$$

$$a_p a_p^+ = [a_p, a_p^+] + a_p^+ a_p$$

$$\boxed{H \doteq \int \frac{d^3\vec{p}}{(2\pi)^3}\,\omega_p\left[a_p^\dagger a_p + \frac{1}{2}(2\pi)^3 \underbrace{\delta^3(0)}_{=\,\infty}\right]}$$

$$\delta^3(\vec{x}) = \begin{cases} \infty, & \vec{x}=0 \\ 0, & \vec{x}\neq 0 \end{cases}$$

1. Are we measuring absolute energy?
→ actually not: only differences

We identify the infinity with the vacuum energy

$$H|0\rangle = \left[\underbrace{\int \frac{d^3\vec{p}}{(2\pi)^3}\,\omega_p}_{\infty}\ \underbrace{\frac{1}{2}(2\pi)^3\delta^3(0)}_{\infty}\right]|0\rangle$$

1^{st} infinity: infinite volume (infrared)

$$(2\pi)^3\delta^3(0) = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3\vec{x}\; e^{i\vec{p}\vec{x}}\Big|_{\vec{p}=0} = V$$

Can be remedied by defining energy density

$$\mathcal{E}_0 = \frac{E_0}{V}$$

2^{nd} infinity: $\mathcal{E}_0 = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\omega_p = \infty$

Due to integrating up to infinite momenta

Remove the vacuum energy

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$$\boxed{H = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \omega_p\, a_p^\dagger a_p}$$

In chosing this form we used <u>normal ordering</u>

$$\langle 0 | \ldots \phi_1 \phi_2 \ldots \phi_n \ldots | 0 \rangle \sim$$

$$\sim \langle 0 | \ldots \{ a_1, a_1^\dagger ; a_2, a_2^\dagger ; \ldots a_n, a_n^\dagger \} \ldots | 0 \rangle$$

$$\downarrow$$

$$\langle 0 | \ldots \{ \underbrace{a_1^\dagger a_2^\dagger \ldots a_n^\dagger}_{\text{all } a^\dagger \text{ to the left}} \; \underbrace{a_1 a_2 \ldots a_n}_{\text{all } a \text{ to the right}} \} \ldots | 0 \rangle$$