Lecture 2   4/11/20

Classical field Theory

Basic object: field $\varphi(\vec{x}, t)$

Classical particle $q_a(t)$

$\downarrow$

$\varphi_{\underline{a}}(\underline{\vec{x}}, \underline{t})$ Both $\vec{x}$ and a are indices of the field

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Example: electromagnetic fields

$\vec{E}(\vec{x}, t)$ $\vec{B}(\vec{x}, t)$

Maxwell's eq's: $\vec{\nabla} \vec{B} = 0$ $\frac{d\vec{B}}{dt} = -\vec{\nabla} \times \vec{E}$

Covariant way: $A^\mu = (\phi, \vec{A})$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

Lagrangian density

$$\mathcal{L} = \mathcal{L}(\varphi(\vec{x}, t), \underbrace{\partial_\mu \varphi(\vec{x}, t)}_{(\dot{\varphi}, \vec{\nabla}\varphi)})$$

$\downarrow$

Lagrangian $L(t) = \int d^3\vec{x}\, \mathcal{L}$

Action $S = \int dt\, L(t) = \int d^4x\, \mathcal{L}$

We adopted some restrictions:

$\rightarrow$ no explicit dep. on x^μ

→ no higher derivatives

Note: higher derivatives in $\mathcal{L}$ are present in effective FT

Require $S = \min$; $\delta S(\varphi_0 \to \varphi_0 + \delta\varphi) = 0$

$$\mathcal{L} \to \mathcal{L}_0 + \delta\mathcal{L}$$

$$\delta\mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)$$

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right]\delta\varphi$$

$$+ \underbrace{\int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)}_{\left.\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right|_{\text{Boundary}}}$$

$\mathbb{R}(3,1)$

$\delta\varphi(\infty) = 0$

↓

Euler-Lagrange Eq.
$$\boxed{\frac{\partial\mathcal{L}}{\partial\varphi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)} = 0}$$

Example: Klein-Gordon field (real scalar)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \varphi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi$$

$$\Rightarrow (\Box + m^2)\varphi = 0 \qquad \underline{\Box = \partial_\mu \partial^\mu}$$

Example: Maxwell's eq. follow from

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A^\nu)(\partial^\mu A_\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = 0 \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = -\partial^\mu A_\nu + g^\mu{}_\nu (\partial_\alpha A^\alpha)$$

$$\downarrow$$

$$0 = -\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = \partial_\alpha \partial^\alpha A_\nu - \partial_\nu (\partial_\alpha A^\alpha)$$

$$= -\partial_\alpha \underbrace{(\partial^\alpha A_\nu - \partial_\nu A^\alpha)}_{= F^\alpha{}_\nu}$$

$$\boxed{\partial_\mu F^{\mu\nu} = 0}$$

$$\underline{\mathcal{L}_{em} = -\frac{1}{4}(F^{\mu\nu} F_{\mu\nu})}$$

Note on Lagrangian: it is local (4)

$$L \sim \int d^3\vec{x} \, \{ \varphi^2(\vec{x},t) \ldots$$

$$\neq \int d^3\vec{x} \int d^3\vec{y} \; \varphi(\vec{x}) \, \varphi(\vec{y})$$

Lorentz invariance of K-G Lagrangian

Lorentz transformation: active

$$\Lambda : \varphi(x) \longmapsto \varphi(\Lambda^{-1} x)$$

$$\partial_\mu \varphi(x) \longmapsto \Lambda^{-1\,\mu}{}_{\nu} \, \partial^\nu \varphi(\Lambda^{-1} x)$$

$$(\partial_\mu \varphi)(\partial^\mu \varphi) \longmapsto \Lambda^{-1\,\mu}{}_{\nu} \, \partial^\nu \varphi(\Lambda^{-1} x)$$

$$\cdot \Lambda^{-1}{}_{\mu}{}^{\alpha} \, \partial_\alpha \varphi(\Lambda^{-1} x)$$

$$= \underbrace{(\Lambda^{-1\,\mu}{}_{\nu}) \, g_{\mu\mu'} (\Lambda^{-1\,\mu'}{}_{\alpha})}_{= g_{\nu\alpha}}$$

$$\cdot \, \partial^\nu \varphi \; \partial^\alpha \varphi$$

$$= (\partial^\nu \varphi(\Lambda^{-1} x))(\partial_\nu \varphi(\Lambda^{-1} x))$$

↳ Maxwell Lagrangian

$$A^\mu \mapsto \Lambda^{\mu\nu} A_\nu (\Lambda^{-1} x)$$ Ex.

Symmetries

Noether theorem:

to every continuous symmetry of $\mathcal{L}$ corresponds a conserved current

$$j^\mu(x):\quad \partial_\mu j^\mu(x) = 0$$

Examples: time transl. $\longleftrightarrow$ energy cons.
central force $\longleftrightarrow$ orbital mom.

Conserved current $\longleftrightarrow$ conserved charge

$$\frac{\partial}{\partial t} j^0 - \vec{\nabla}\vec{j} = 0 \Rightarrow Q = \int d^3\vec{x}\, j^0$$

The charge is conserved locally
$$\frac{\partial Q}{\partial t} = \int d^3\vec{x}\,(\vec{\nabla}\vec{j}\,)$$
$$= \int_S (\vec{j}\,'\cdot d\vec{S}\,)$$

Proof Consider infinitesimal transf.

L is symmetric under $\varphi_a \to \varphi_a + \delta\varphi_a$

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} \underset{\text{symmetric}}{\longleftrightarrow} \delta\mathcal{L} = \partial_\mu j^\mu$$

$$\delta\mathcal{L} = \delta\mathcal{L}(\varphi_a + \delta\varphi_a,\ \partial_\mu\varphi_a + \partial_\mu\delta\varphi_a \ldots)$$
$$= \delta\varphi_a \frac{\partial\mathcal{L}}{\partial\varphi_a} + (\partial_\mu\delta\varphi_a)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\delta\varphi_a)}$$

$$= \partial_\mu \left(\delta\varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right)$$

$$+ \delta\varphi_a \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right]}_{\simeq 0} \quad \text{E-L eq.}$$

$$= \partial_\mu j^\mu$$

$$\boxed{\partial_\mu J^\mu = \partial_\mu \left(\delta\varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} - j^\mu \right) = 0}$$

J^μ — conserved current.

Example 1 inf. translation

$$x^\mu \to x^\mu - \underline{\varepsilon}^\mu$$

$$\phi_a \mapsto \phi_a(x+\varepsilon) = \phi_a + \varepsilon^\nu \partial_\nu \phi$$

$$\mathcal{L} \to \mathcal{L} + \varepsilon^\nu \partial_\nu \mathcal{L}$$

$$\hookrightarrow T^{\mu\nu} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial^\nu \phi_a - g^{\mu\nu} \mathcal{L} \right)$$

Energy – momentum $T^{\mu\nu}$: $\partial_\mu T^{\mu\nu} = 0$

$$E = \int d^3\vec{x}\, T^{00} \qquad \vec{P}^i = \int d^3\vec{x}\, T^{0i}$$

From K-G Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2$$

$$\hookrightarrow T^{\mu\nu} = (\partial^\mu \varphi)(\partial^\nu \varphi) - g^{\mu\nu}\mathcal{L}$$

$$E = \int d^3\vec{x}\left[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2\right]$$

$$\vec{P} = \int d^3\vec{x}\ \dot{\varphi}\,\vec{\nabla}\varphi$$

4D angular mom.

Infinitesimal Lorentz transf.

$$\Lambda^{\mu\nu} = g^{\mu\nu} + \omega^{\mu\nu} \qquad |\omega| \ll 1$$

$$\Lambda^{\mu\alpha}\Lambda^{\nu}{}_{\alpha} = g^{\mu\nu} = (g^{\mu\alpha} + \omega^{\mu\alpha})(g^{\nu}{}_{\alpha} + \omega^{\nu}{}_{\alpha})$$

$$= g^{\mu\nu} + \underbrace{\omega^{\mu\nu} + \omega^{\nu\mu}}_{\omega^{\mu\nu} + \omega^{\nu\mu} = 0} + \cancel{\omega^{\mu\alpha}\omega^{\nu}{}_{\alpha}}$$

$$\varphi(x) \to \varphi'(x) = \varphi(\Lambda^{-1}x) = \varphi(x^\mu - \omega^{\mu\nu}x_\nu)$$

$$= \varphi(x) - \omega^{\mu\nu}x_\nu \partial_\mu \varphi$$

$$\underline{\delta\varphi = -\omega^{\mu\nu}x_\nu\partial_\mu\varphi}$$

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(-\omega^{\alpha\nu}x_\nu\partial_\alpha\varphi_a)\right)$$

$$= -\partial_\mu \left(\omega^{\mu\nu} x_\nu \mathcal{L} \right) \doteq \partial_\mu j^\mu$$ 8

$$\mapsto \omega^{\mu\nu} \underbrace{\partial_\mu x_\nu}_{g_{\mu\nu}} = \omega^{\mu\nu} g_{\mu\nu} = 0$$

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \omega^{\alpha\beta} x_\beta \partial_\alpha \varphi + \omega^{\mu\nu} x_\nu \mathcal{L}$$

$$= -\omega^{\alpha\beta} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} x_\beta \partial_\alpha \varphi - g^\mu{}_\alpha x_\beta \mathcal{L} \right]$$

$$= -\omega^{\alpha\beta} T^\mu{}_\alpha x_\beta$$

$$\hookrightarrow J^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}$$

$$\partial_\mu J^{\mu\alpha\beta} = 0$$ 6 conserved currents

$\alpha, \beta = 1,2,3$ $$Q^{ij} = \int d^3\vec{x} \left[x^i T^{0j} - x^j T^{0i} \right]$$
$$= \varepsilon^{ijk} L^k \qquad L = [\vec{x} \times \vec{P}]$$

$\alpha = 0$ $$Q^{0i} = \int d^3\vec{x} \left(x^0 T^{0i} - x^i T^{00} \right)$$

$$\frac{dQ^{0i}}{dt} = \int d^3\vec{x} \left(T^{0i} + \frac{\partial}{\partial t} T^{0i} - \frac{\partial x^i}{\partial t} T^{00} \right)$$

$$= \underset{\text{const.}}{P^i} + \underset{0}{\frac{\partial P^i}{\partial t}} - \frac{\partial}{\partial t} \underbrace{\int d^3\vec{x}\, x^i T^{00}}_{\text{center of en.}}$$

Newton's 1st law

Hamiltonian formalism

Conjugate momentum: $\varphi_a \longrightarrow \pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_a}$

Hamiltonian density $\underline{\mathcal{H} = \pi_a \dot{\varphi}_a - \mathcal{L}}$

Rea scalar field $\mathcal{L} = \frac{1}{2}\left(\dot{\varphi}^2 - (\vec{\nabla}\varphi)^2 - V(\varphi)\right)$

$$\underbrace{\mathcal{H} = \frac{1}{2}\left(\dot{\varphi}^2 + (\vec{\nabla}\varphi)^2 + V(\varphi)\right)}_{\text{Total energy}}$$

Eqs. of motion $\quad \dot{\varphi} = \frac{\partial \mathcal{H}}{\partial \pi} \qquad \dot{\pi} = \frac{\partial \mathcal{H}}{\partial \varphi}$

Important: $\mathcal{H}$ not L. invariant

$\mathcal{L}$ is

Poisson's brackets:

$$\{U, V\} = \sum_{i=1}^{N} \left[\frac{\partial U}{\partial q_i} \frac{\partial V}{\partial p_i} - \frac{\partial U}{\partial p_i} \frac{\partial V}{\partial q_i} \right]$$

$$\{q_a, q_b\} = \{p_a, p_b\} = 0$$

$$\{q_a, p_b\} = \delta_{ab}$$

Evolution $\quad \frac{dU}{dt} = \{U, H\}$